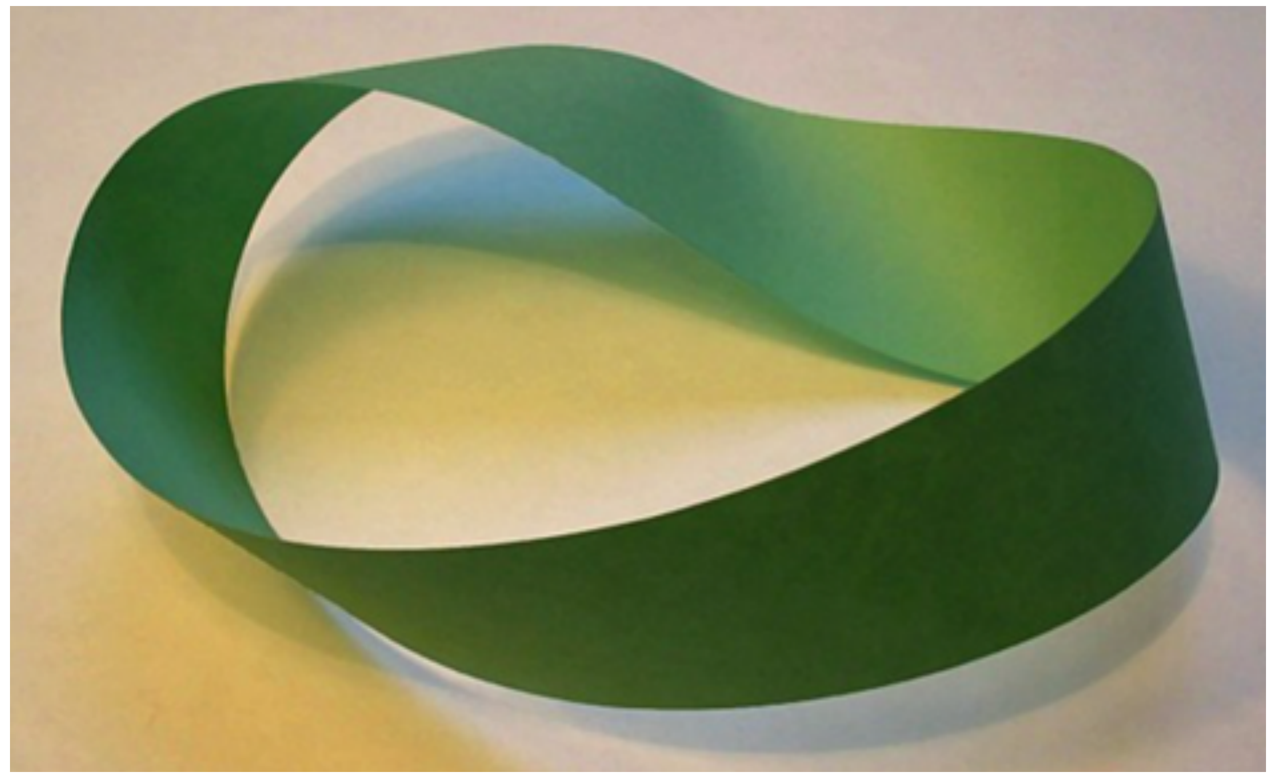

The Mystique of Quantum Spin
From Rodrigues, Hamilton…to Pauli, Dirac and Hestenes

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All things in universe are made of **Elementary particles - called spin-1/2 particles. They are point particles**

Three fundamental attributes of these particles:

Mass and charge : Scalar quantities

"SPIN" - intrinsic angular momentum EXACTLY equal to- $\frac{1}{2}\hbar$ - called 1/2 integer spin

In quantum world, these particles are described by wave functions - state of the particle.

Mysterious thing about these wave functions is: under rotation by 360 degrees, they change SIGN

$\psi \rightarrow -\psi$ as $\theta \rightarrow \theta + 2\pi$.

That is: Particles with half-integer spins, do not recover their original posture or state after one full rotation. The need to complete two full rotations before returning to their original state, something that is reminiscent of

Quantities that change sign under rotation by 2π are not scalars, vectors or tensors

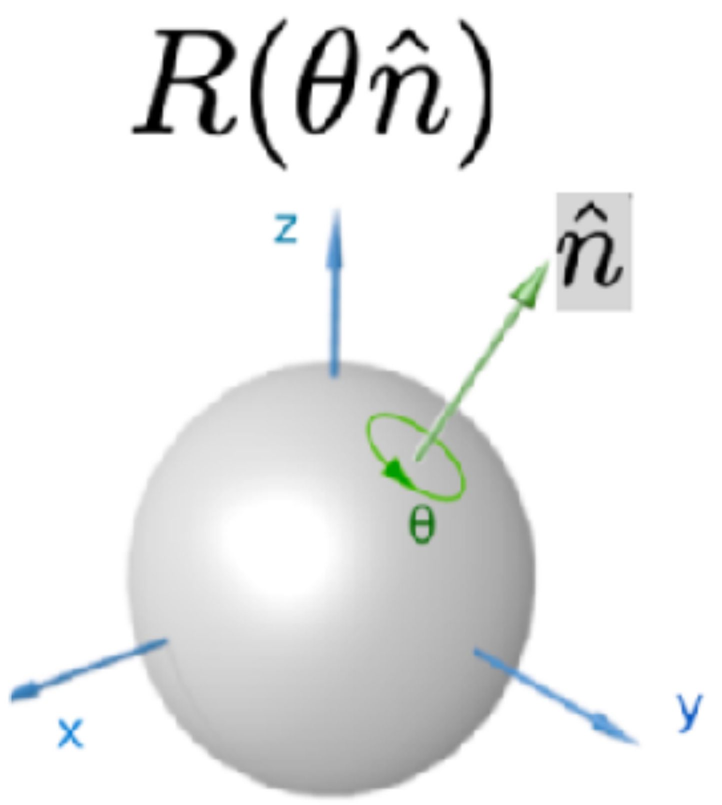
They are called **SPINORS** .

Why talk about spin & spinors in ``Quaternion session" ? Because **spins-1/2 are quaternions** in disguise

Euler: 1775: Euler's Rotation Theorem or Fixed Point Theorem

In 3D, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.

Let us represent this rotation $R(\theta\hat{n})$.



He showed that the composition of two rotations is also a rotation.

$$R(\gamma\hat{n}) = R(\alpha\hat{l})R(\beta\hat{m})$$

Olinde Rodrigues: 1840: Product formula

Gives axis and angle of final rotation in terms of axes and angles of two consecutive rotations.

$$\begin{aligned}\cos\frac{\gamma}{2} &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\hat{l}\cdot\hat{m} \\ \sin\frac{\gamma}{2}\hat{n} &= \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\hat{l} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\hat{m} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\hat{l}\times\hat{m}\end{aligned}$$

"Proof of the formula: "arXiv:2211.08333" Translation by Richard Friedberg: Nov: 2022.

Rotation $R(\theta\hat{n})$ is described by four parameters:

$$q_0 = \cos\tfrac{\theta}{2}, \quad \vec{q} = \sin\tfrac{\theta}{2}\ \hat{n}, \quad \hat{n} = (n_x, n_y, n_z)$$

Angles of rotation appear in terms of half-angles

Hamilton: **1843** Discovers Quaternions

Reincarnation of Quaternion

Pauli matrices obey quaternion algebra

Do complex numbers $x+iy$ that describe 2D rotation have generalization to triplets $x+iy+jk$ to describe 3D rotation?

REVIEW: In 2D, vector $\vec{v} \doteq x+iy \equiv re^{i\theta}$

To rotate $\vec{v}$ by α, multiply the corresponding complex number by $e^{i\alpha}$

$$re^{i(\alpha+\theta)} = e^{i\alpha} re^{i\theta}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_i^2 = 1, \quad \sigma_i\sigma_j + \sigma_j\sigma_i = 0, \; i \neq j., \quad \sigma_i\sigma_j = \epsilon_{ijk} i\sigma_k$$

$$\hat{I} \to i\sigma_z, \quad \hat{J} \to i\sigma_y, \quad \hat{K} \to i\sigma_x$$

Eureka Moment: Need Quadruplets not Triplets:

$(x+iy) \to (s + x\hat{I} + y\hat{J} + z\hat{K})$ – *quaternions*

$(\hat{I}, \hat{J}, \hat{K})$ obey following rules:

$$\hat{I}^2 = \hat{J}^2 = \hat{K}^2 = -1$$

$$\hat{I}\hat{J} = -\hat{J}\hat{I}, \quad \hat{J}\hat{K} = -\hat{K}\hat{J}, \quad \hat{K}\hat{I} = -\hat{I}\hat{K}$$

$$\hat{I}\hat{J} = \hat{K}, \quad \hat{J}\hat{K} = \hat{I}, \quad \hat{K}\hat{I} = \hat{J}$$

Rotation $R(\theta\hat{n})$

Unit Quaternion

$$R = \cos\frac{\theta}{2} + \hat{\mathbf{n}}\sin\frac{\theta}{2} \equiv e^{-\hat{\mathbf{n}}\frac{\theta}{2}}, \quad \hat{\mathbf{n}} = \mathbf{n_x}\hat{\mathbf{I}} + \mathbf{n_y}\hat{\mathbf{J}} + \mathbf{n_z}\hat{\mathbf{K}},$$

$$\vec{v'} = e^{-\hat{\mathbf{n}}\frac{\theta}{2}} \; \vec{v} \; e^{\hat{\mathbf{n}}\frac{\theta}{2}} \equiv R \; \vec{v} \; \tilde{R},$$

$$R = \cos\frac{\theta}{2} + \hat{\mathbf{n}}\sin\frac{\theta}{2}, \quad \tilde{R} = \cos\frac{\theta}{2} + \hat{\mathbf{n}}\sin\frac{\theta}{2}$$

Note: Half-angles

Geometric Algebra

Hermann Grassmann (**1807-1877**) :

William Kingdon Clifford **(1845–1879)** :

Introduce "geometric product" as

$$e_1e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 \equiv \frac{1}{2}(e_1e_2 + e_2e_1) + \frac{1}{2}(e_1e_2 - e_2e_1)$$

Symmetrized product is a scalar

$$(a+b)^2 = (a+b)(a+b) = a^2 + (ab+ba) + b^2$$

$$ab + ba = (a+b)^2 - a^2 - b^2, ...is\ a\ scalar$$

$e_i^2 = 1, \quad e_ie_j = -e_je_i, \quad i \neq$

1 (scalar)

e_1, e_2, e_3 (vectors)

$(e_1e_2 \equiv e_{12}, e_2e_3 \equiv e_{23}, e_3e_1 \equiv e_{31})$ (bivectors - area elements)

$e_1e_2e_3 \equiv e_{123} \equiv i_g$ (trivector , volume element- pseudosclar)

$$(e_1e_je_3)e_1 = e_2e_3 = i_ge_1$$

$$(e_1e_je_3)e_2 = e_3e_1 = i_ge_2$$

$$(e_1e_je_3)e_3 = e_1e_2 = i_ge_3$$

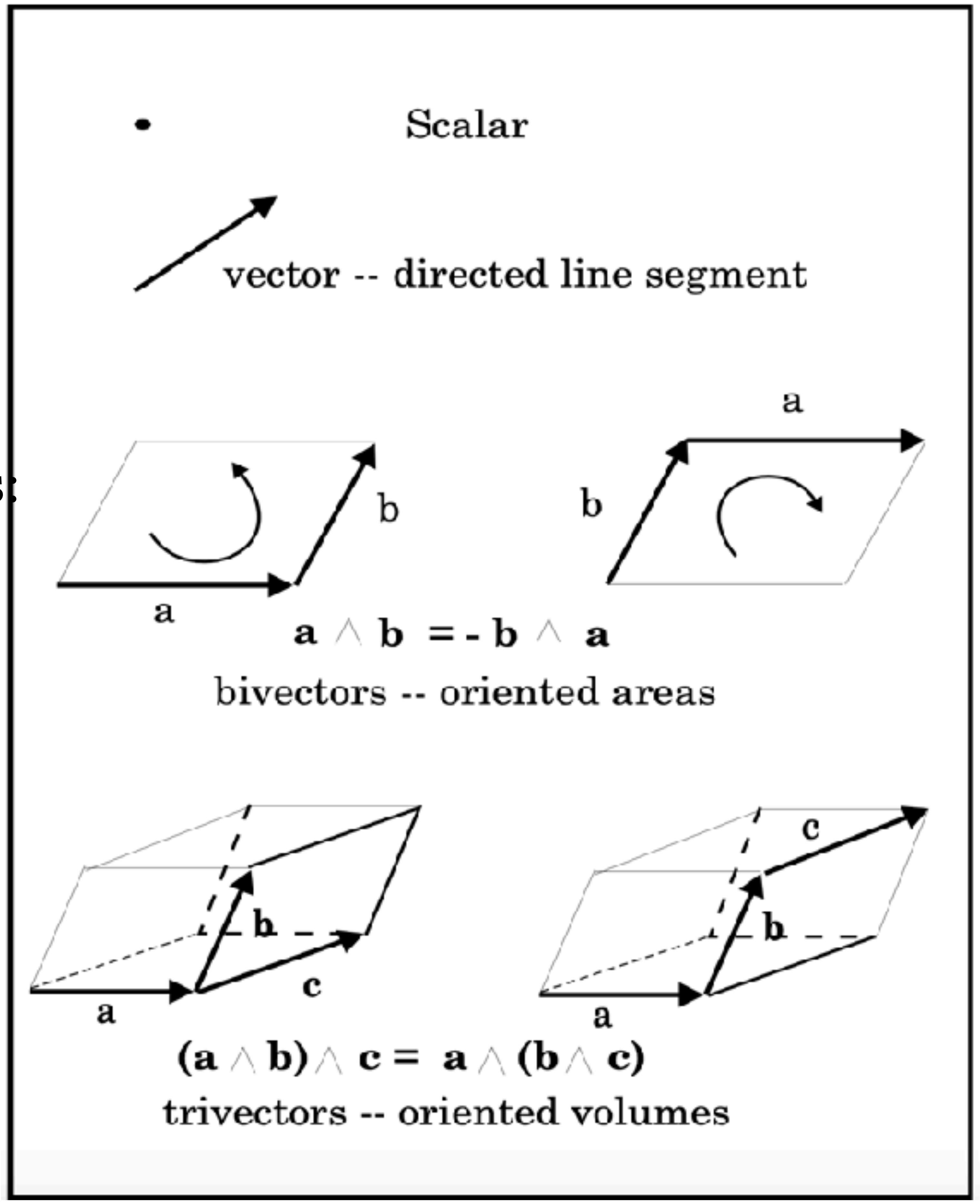
Consider three orthonormal vectors:

Surprise

In 3D, Clifford algebra is Pauli Algebra

$$\sigma_i\sigma_j = \epsilon_{ijk}i\sigma_k, \quad i \neq j$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Where are Quaternion?

$$I = e_2 e_3, \quad J = e_3 e_1, \quad K = e_1 e_2,$$

Unit quaternions: $R = \cos\frac{\theta}{2} + \hat{\mathbf{n}}\sin\frac{\theta}{2}$ - bivectors

$$\begin{aligned} R &= \cos\frac{\theta}{2} + (n_x e_2 e_3 + n_y e_3 e_1 + n_z e_1 e_2)\sin\frac{\theta}{2} \\ &= \cos\frac{\theta}{2} + i_g(n_x e_1 + n_y e_2 + n_z e_3)\sin\frac{\theta}{2} \\ &= \cos\frac{\theta}{2} + i_g\hat{n}\cdot\hat{e}\sin\frac{\theta}{2} \doteq e^{i_g\hat{n}\cdot\hat{e}\frac{\theta}{2}} \end{aligned}$$

Bivectors are “spinors” - this links spinors with rotations explicitly.

Summarizing GA

(1) Instead of scalars and vectors, we have multivectors. That is, scalars, vectors, and bivectors which are of different ranks are part of the same algebra. Bivectors and trivectors are of rank two and three respectively, and represent two different geometrical entities.

(2) Even-ranked multivectors form their own own subalgebra. Bivectors are very important in understanding spin.

(3) The key feature of multivector algebra is the rule for multiplying vectors. While the product ab of two vectors is not itself a vector, it is nevertheless composed of quantities with geometrical significance. Geometric product of two vectors describes a degree of commutativity of two vectors. Commutativity means that the vectors are collinear. Anticommutativity means that they are orthogonal.

(4) Unlike vector products, geometric products define a closed algebra. For example, given three vectors $(\vec{a}, \vec{b}, \vec{c})$,

$\vec{a} \cdot (\vec{b} \times \vec{c}) \neq (\vec{a} \cdot \vec{b}) \times \vec{c}$, but $\vec{a} \cdot (\vec{b} \wedge \vec{c}) = (\vec{a} \cdot \vec{b}) \wedge \vec{c}$

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$, but $\vec{a} \wedge (\vec{b} \wedge \vec{c}) = (\vec{a} \wedge \vec{b}) \wedge \vec{c}$

$$a \wedge b = i_g a \times b$$

$$a \wedge b = i_g a \times b$$

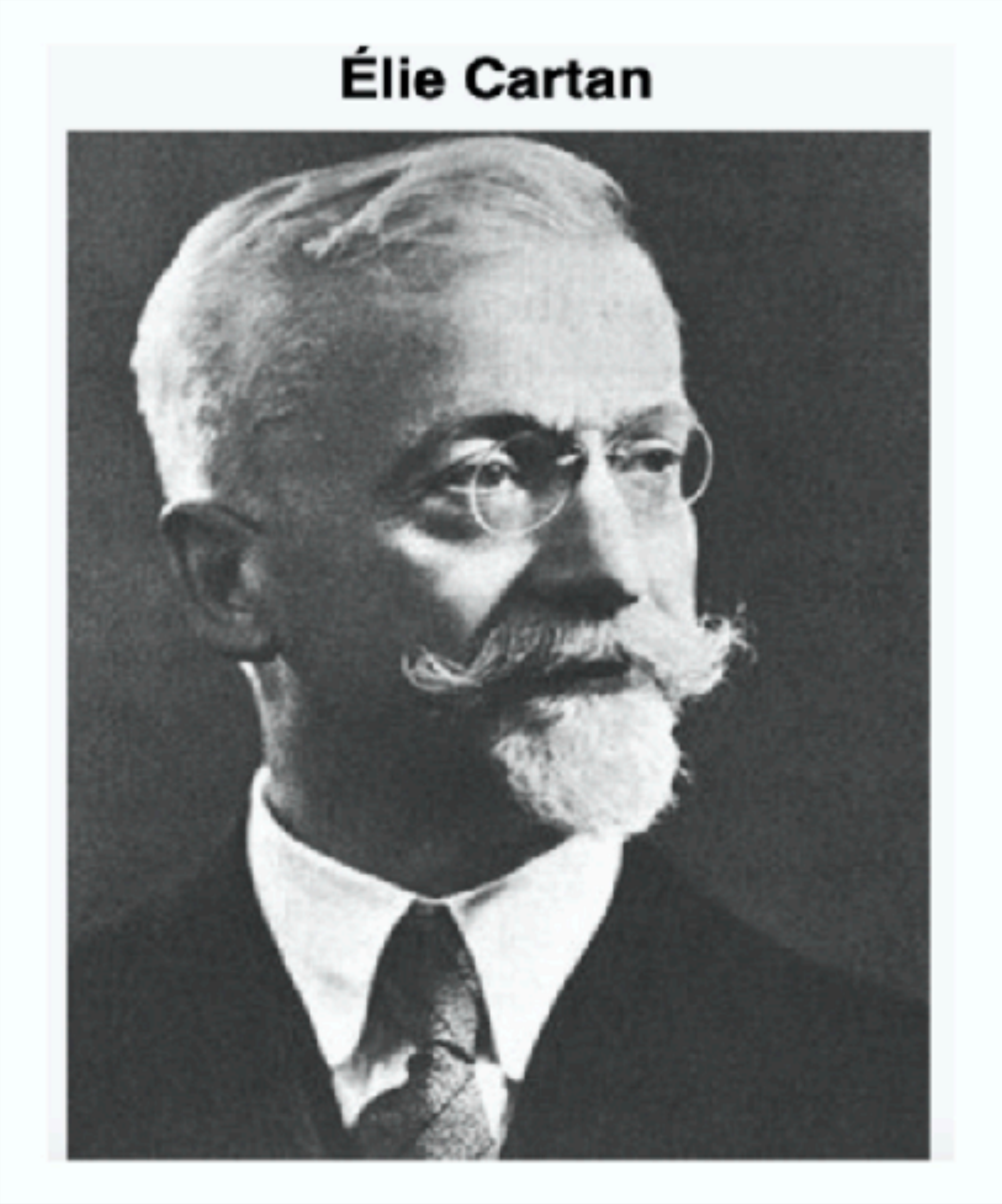
Example:

$$\vec{a} = e_1 + 2e_2 + 3e_3, \quad \vec{b} = 4e_1 + 5e_2 + 6e_3, \quad \vec{a} \times \vec{b} = -3e_1 - 6e_2 - 3e_3$$

$$\begin{aligned}
\vec{a} \wedge \vec{b} &= (e_1 + 2e_2 + 3e_3) \wedge (4e_1 + 5e_2 + 6e_3) \\
&= 5e_{12} + 6e_{13} + 8e_{21} + 12e_{23} + 12e_{31} + 15e_{32} \\
&= 3e_{21} + 6e_{31} + 3e_{32} \\
&= -3e_{12} - 6e_{13} - 3e_{23} \\
&= e_{123}(-3e_3) + e_{132}(-6e_2) + e_{231}(-3e_1) \\
&= e_{123}(-3e_3 - 6e_2 - 3e_1)
\end{aligned}$$

Spin was discovered in physics 1925: But before that

1913: Cartan develops Theory of Spinors as abstract mathematical objects



French Mathematician:
1865-1951

"One of the principal purposes of (my) work is to develop systematically the theory of spinors, by giving an entirely geometric definitions of these mathematical entities: thanks to the geometric origin, the matrices used by physicists in quantum mechanics turn up by themselves.
"... Élie Cartan (1938)

Consider a vector $\vec{x} = (x_1, x_2, x_3)$ in 3D (complex) Euclidean space, of zero length

That is $x^2 = x_1^2 + x_2^2 + x_3^2 = 0$.

Example: Given two orthonormal vectors $\vec{p}$ and $\vec{q}$, define $\vec{x} = \vec{p} + i\vec{q}, \quad p^2 = q^2 = 1$ & $\vec{p} \cdot \vec{q} = 0$.

$x^2 = (\vec{p} + i\vec{q})^2 = p^2 - q^2 + 2i\vec{p} \cdot \vec{q} = 0$.

Associate two numbers (ξ_1, ξ_2) to this vector:

$$x_1 = \xi_1^2 - \xi_2^2, \quad x_2 = i(\xi_1^2 + \xi_2^2), \quad x_3 = -2\xi_1\xi_2$$

$$\xi_1 = \pm[\frac{1}{2}(x_1 - ix_2)]^{\frac{1}{2}}, \quad \xi_2 = \pm[-\frac{1}{2}(x_1 + ix_2)]^{\frac{1}{2}}$$

$\vec{x}$ transform as vectors under rotation: ξ_1 and ξ_2 change sign under 2π roattion.

The pair $\hat{\xi} = (\xi_1, \xi_2)$ is defined as a **spinor**.

Added bonus: Pauli matrices emerge

$$\frac{\xi_1}{\xi_2} = \sqrt{\frac{x_1 - ix_2}{-x_1 - ix_2}}, = \sqrt{\frac{(x_1 - ix_2)^2}{-(x_1 + ix_2)(x_1 - ix_2)}} = -\frac{x_1 - ix_2}{x_3} = \frac{x_3}{x_1 + ix_2}$$

$$x_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + x_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \equiv (\vec{x} \cdot \vec{\sigma})\ \hat{\xi} = 0,$$

How Cartan's spinor transform under rotation

$$x_1 = \xi_1^2 - \xi_2^2, \quad x_2 = i(\xi_1^2 + \xi_2^2), \quad x_3 = -2\xi_1\xi_2$$

$$\xi_1 = \pm[\frac{1}{2}(x_1 - ix_2)]^{\frac{1}{2}}, \quad \xi_2 = \pm[-\frac{1}{2}(x_1 + ix_2)]^{\frac{1}{2}}$$

$$\begin{aligned}
(\xi_1')^2 &= \frac{1}{2}(x_1' - ix_2') \\
&= \frac{1}{2}[(R_{11} - iR_{21})(\xi_1^2 - \xi_2^2) + i(R_{12} - iR_{22})(\xi_1^2 + \xi_2^2) - 2\xi_1\xi_2(R_{13} + R_{23})] \\
&= \frac{1}{2}[(R_{11} + R_{22} - iR_{21} + iR_{12})\xi^2 - 2(R_{13} - iR_{23})\xi_1\xi_2 + (-R_{11} + R_{22} + iR_{21} + iR_{12}
\end{aligned}$$

Orthogonality of $R(\theta\hat{n})$ gives r.h.s is a perfect square: ξ_1' and ξ_2' depend linearly on ξ_1 and ξ_2.

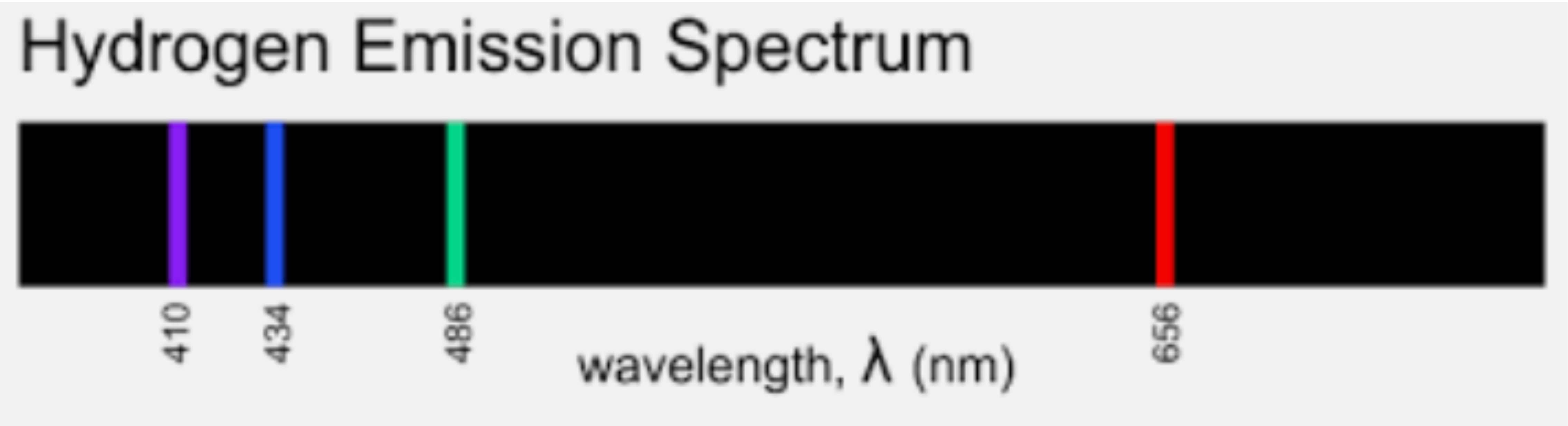
Check: planar rotation about z-axis where the matrix R is given by,

$$R(\theta\hat{n}) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

$$\begin{aligned}
(\xi_1')^2 &= \frac{1}{2}(x_1' - ix_2') \\
&= \frac{1}{2}[(\cos\theta\ x_1 - \sin\theta\ x_2) - i(\sin\theta\ x_1 + \cos\theta\ x_2)] \\
&= \frac{1}{2}e^{i\theta}(x_1 - ix_2) = e^{-i\theta}\xi_i^2
\end{aligned}$$

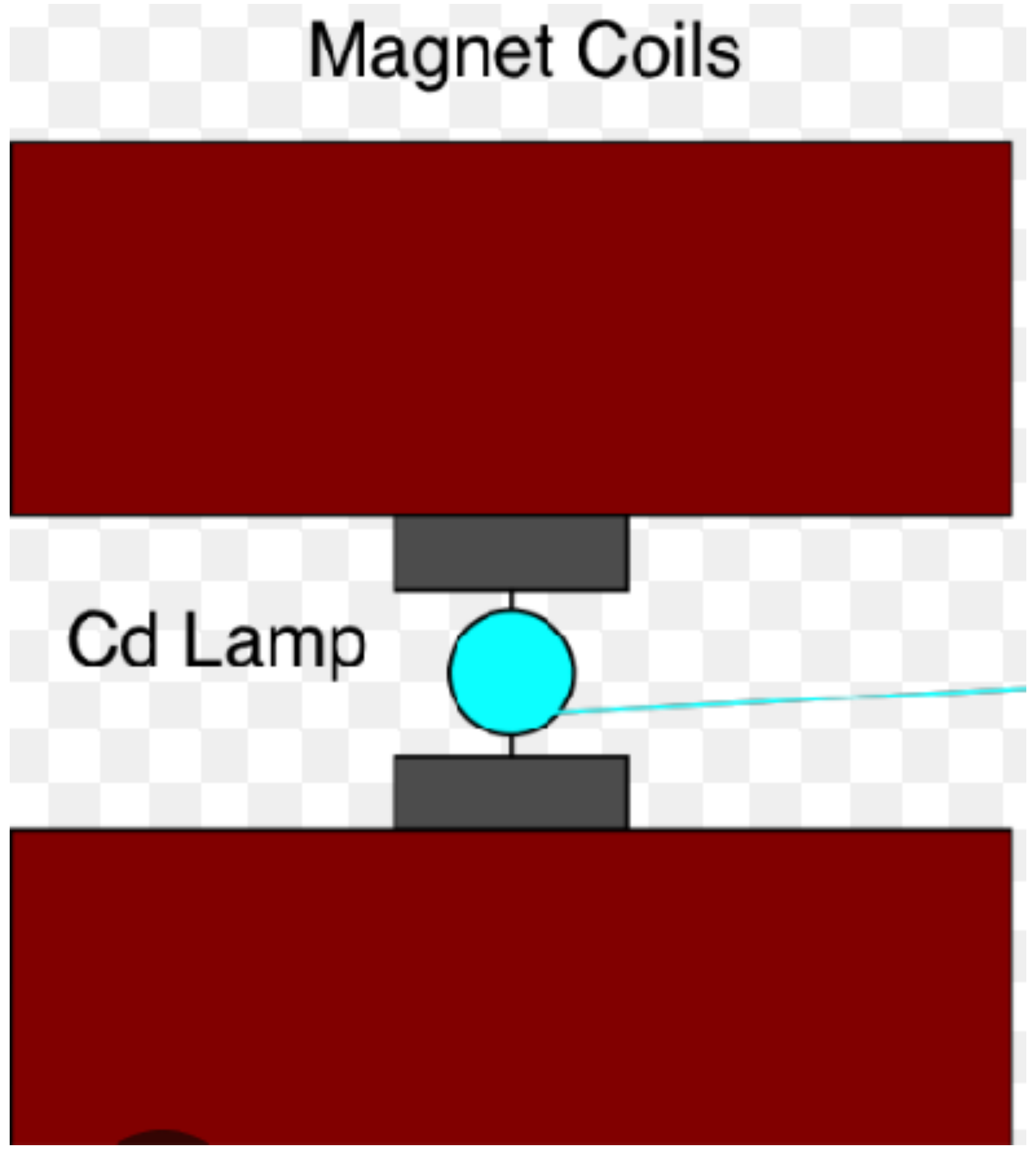
$$\begin{bmatrix} \xi_1' \\ \xi_2' \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = U \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

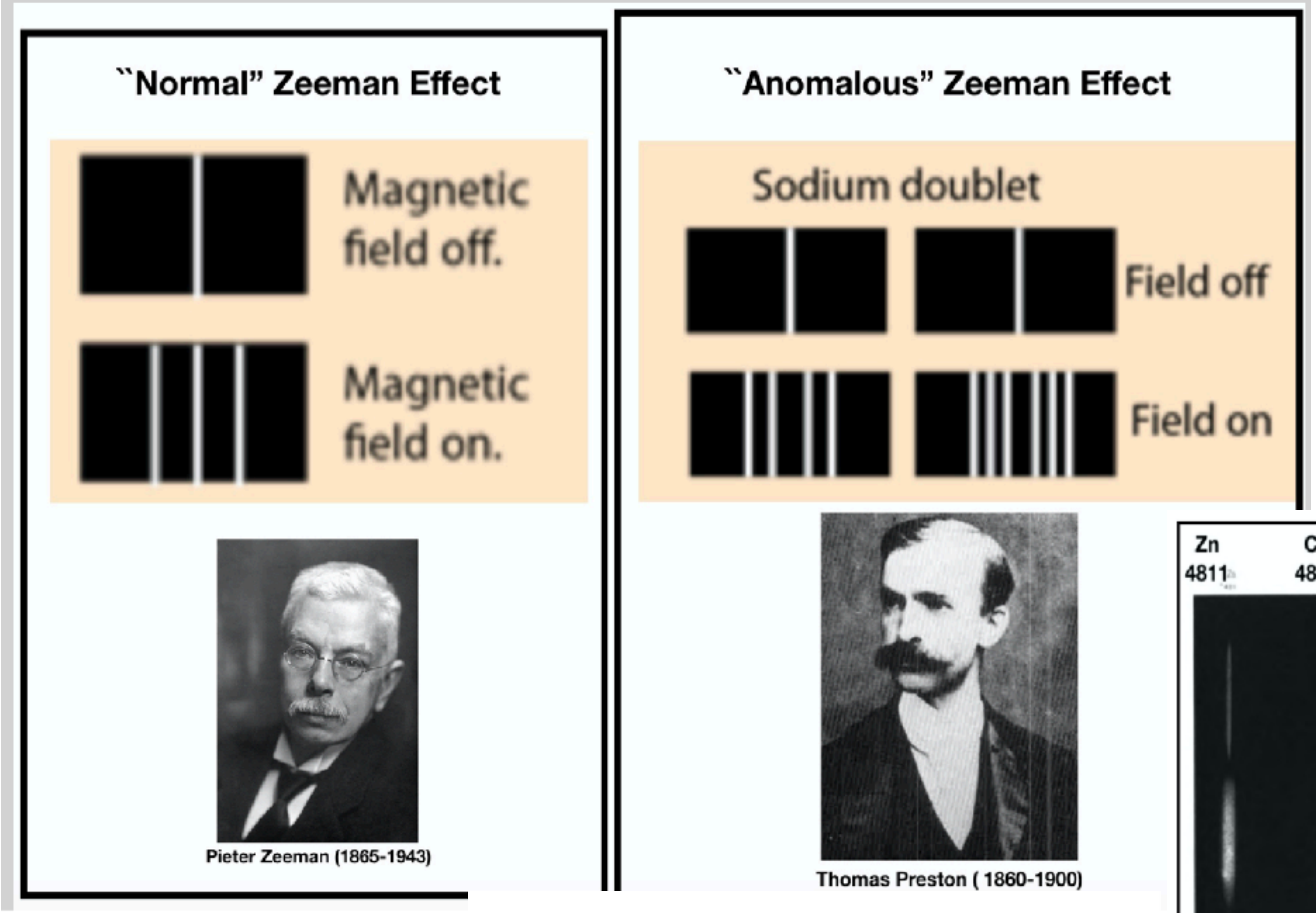
Note: Half-angles Change of sign under 360 degree rotation

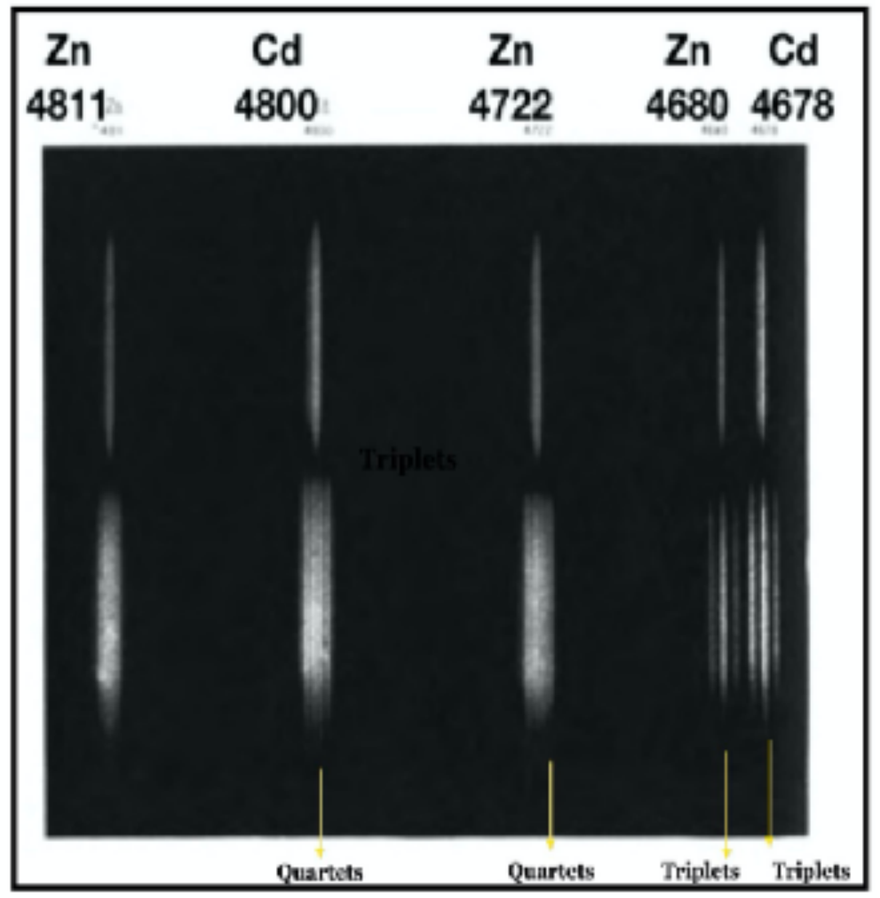


1896-1897: First nail in the coffin of Classical Physics
(note: electron was discovered in Oct 1897)

What happens to spectral lines in presence of magnetic field ???

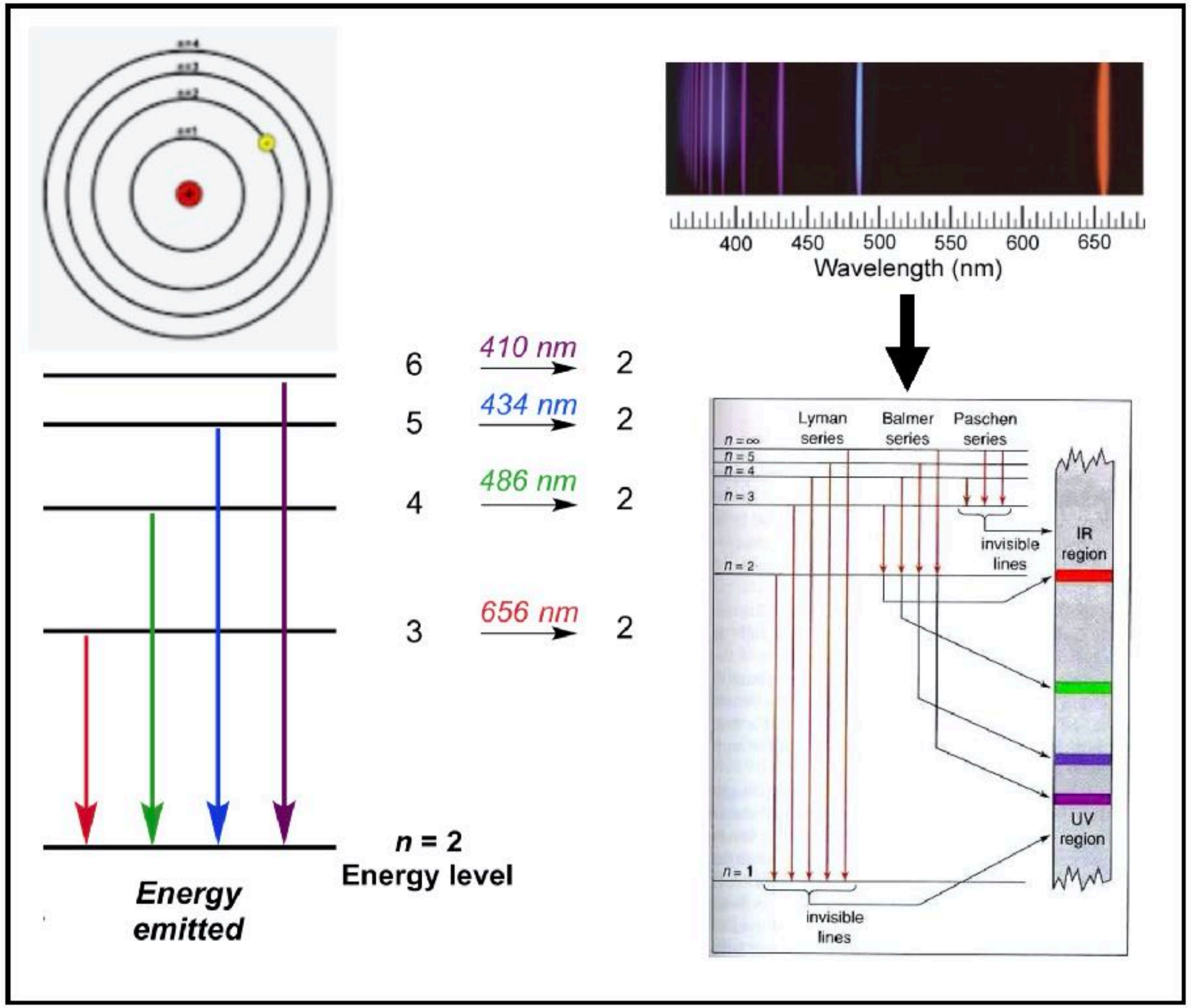


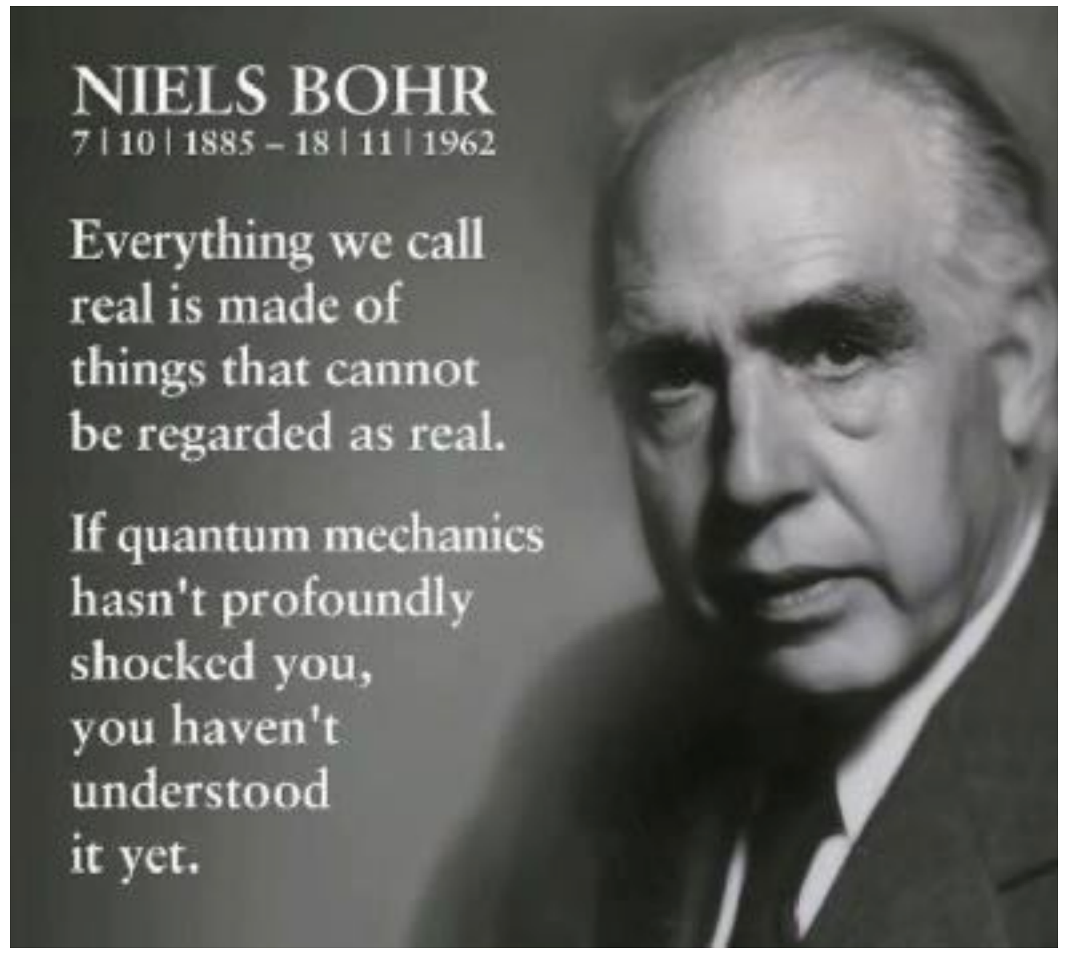


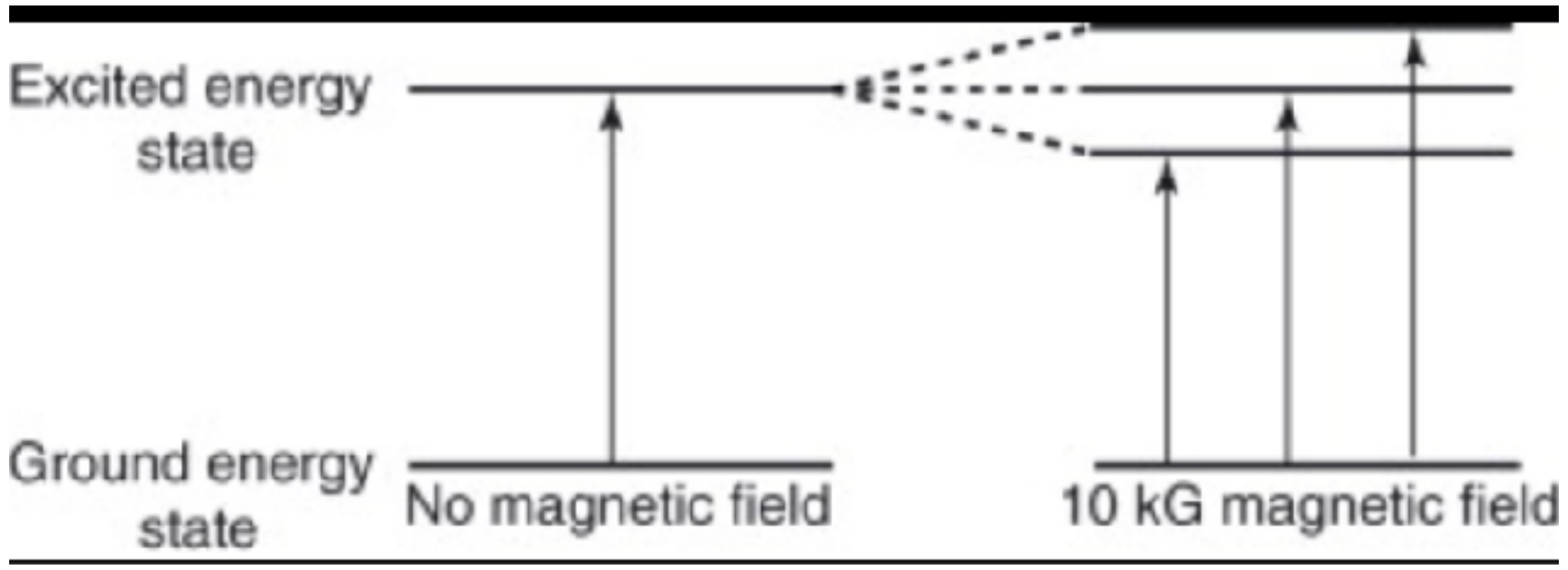


$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_\pm = \omega_0 \pm \frac{eB}{2mc}$$

Bohr Model- 1913

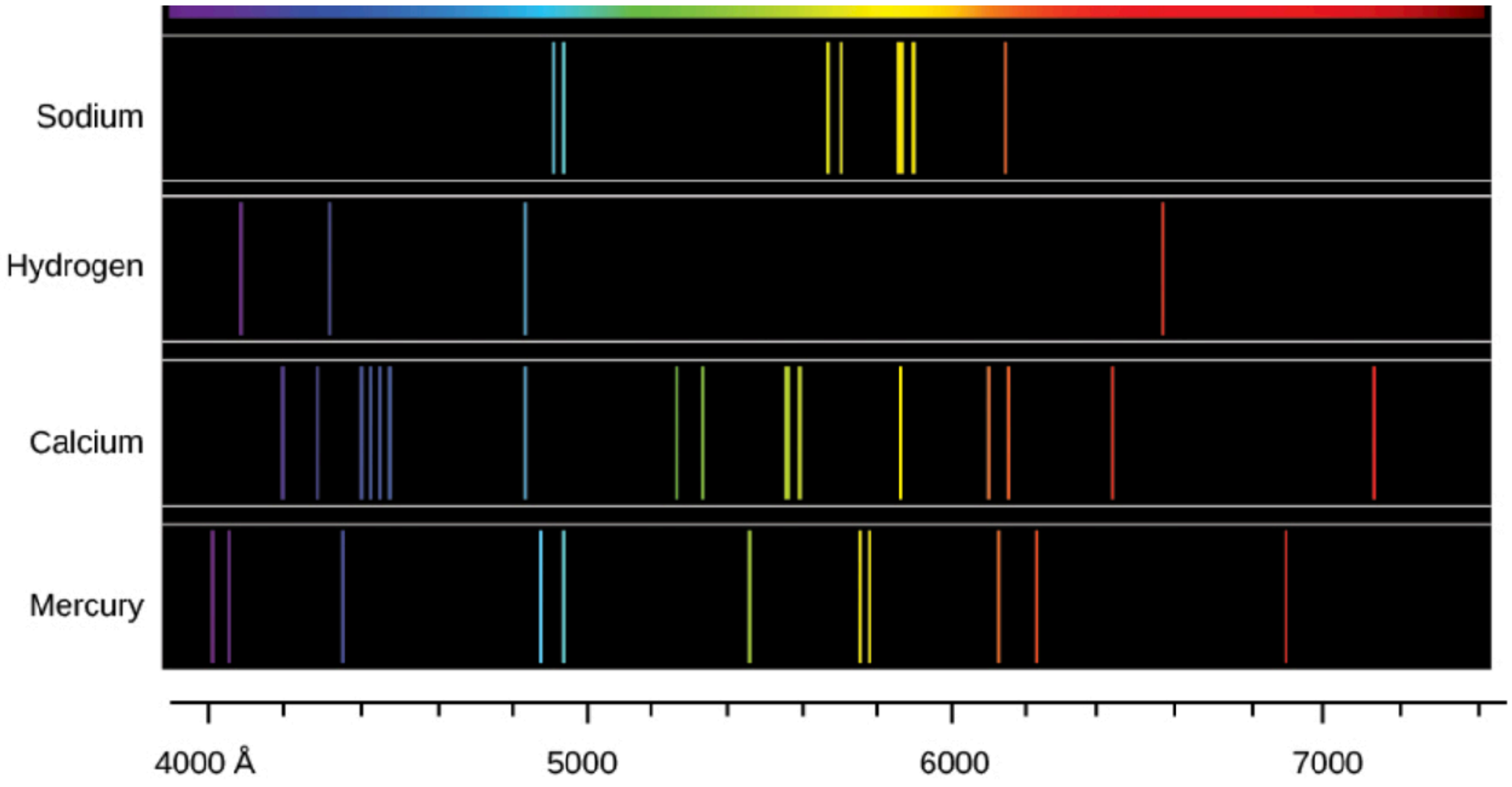






Zeeman effect can also be explained
in quantum theory

More Anomalous splittings observed in atoms with more than one electron



Need to look beyond Bohr Model

1925: Goudsmit & Uhlenbeck propose a model of spinning Electron

Well, that is a nice idea, though it may be wrong. But you don't yet have a reputation, so you have nothing to lose" … Ehrenfest

1925-1927:

Austrian physicist:
(1887-1961)

Erwin Schrödinger

$$E = \frac{p^2}{2m} + V(\vec{r}, t)$$

$$E \rightarrow i\hbar\frac{\partial}{\partial t}, \quad \vec{p} \rightarrow \frac{\hbar}{i}\nabla$$

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}, t)\right]\Psi(\vec{r}, t)$$

How to incorporate spin ?

$$S_z \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial \phi} \; ??$$

Wolfgange Pauli
(1900-1958)

$$\Psi(x) \rightarrow \Psi(x, s_z) = \begin{bmatrix} \Psi(x, \frac{1}{2}) \\ \Psi(x, -\frac{1}{2}) \end{bmatrix} \equiv \begin{bmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{bmatrix} \longrightarrow \textit{Pauli Spinor}$$

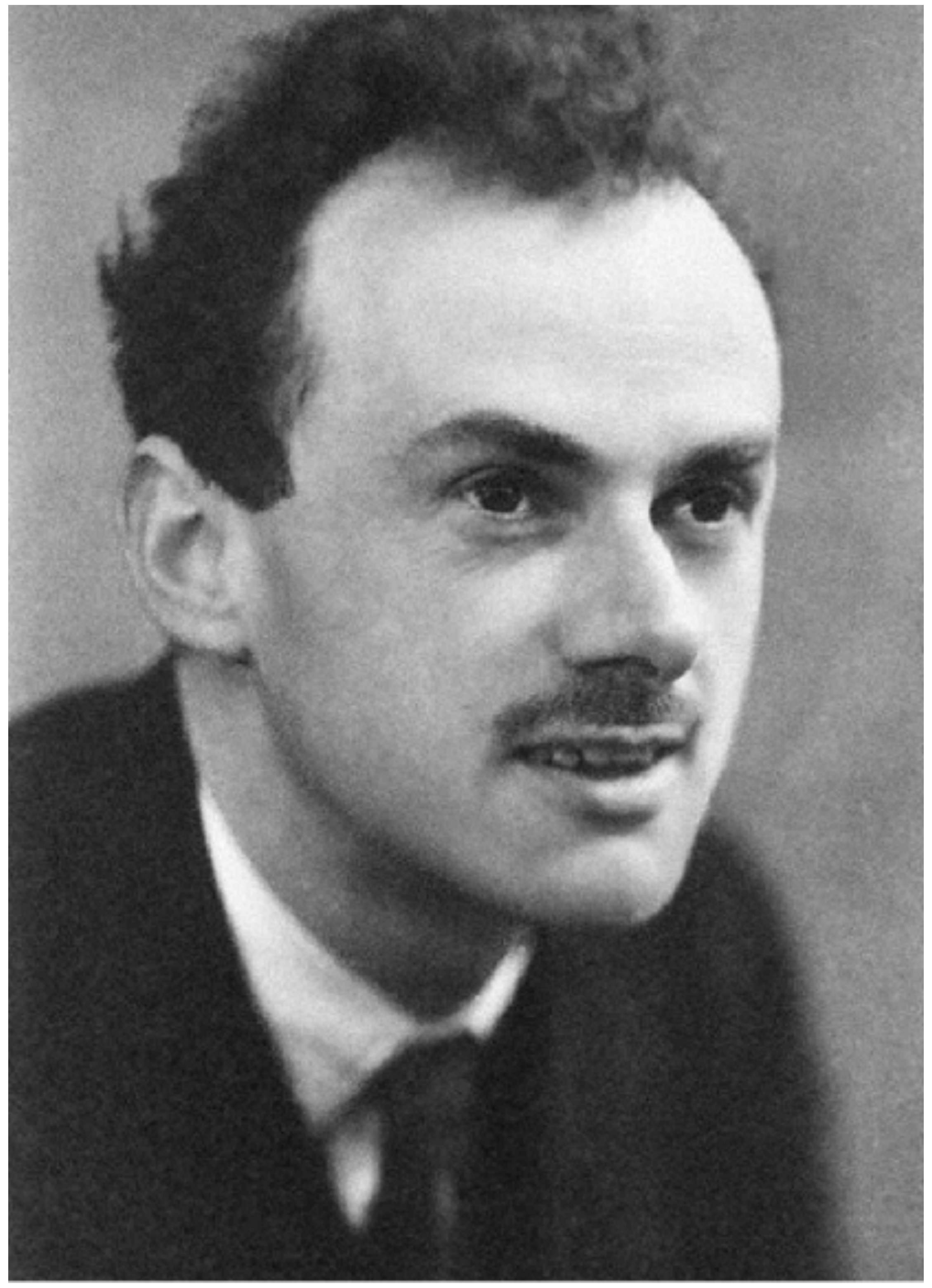
$$i\hbar\partial_t \begin{bmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{bmatrix} = H_0 \begin{bmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{bmatrix} - \frac{e\hbar}{mc}\vec{s}\cdot\vec{B}\begin{bmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{bmatrix}$$

$$\vec{S} = (S_x, S_y, S_z) = \hbar\vec{s}.$$

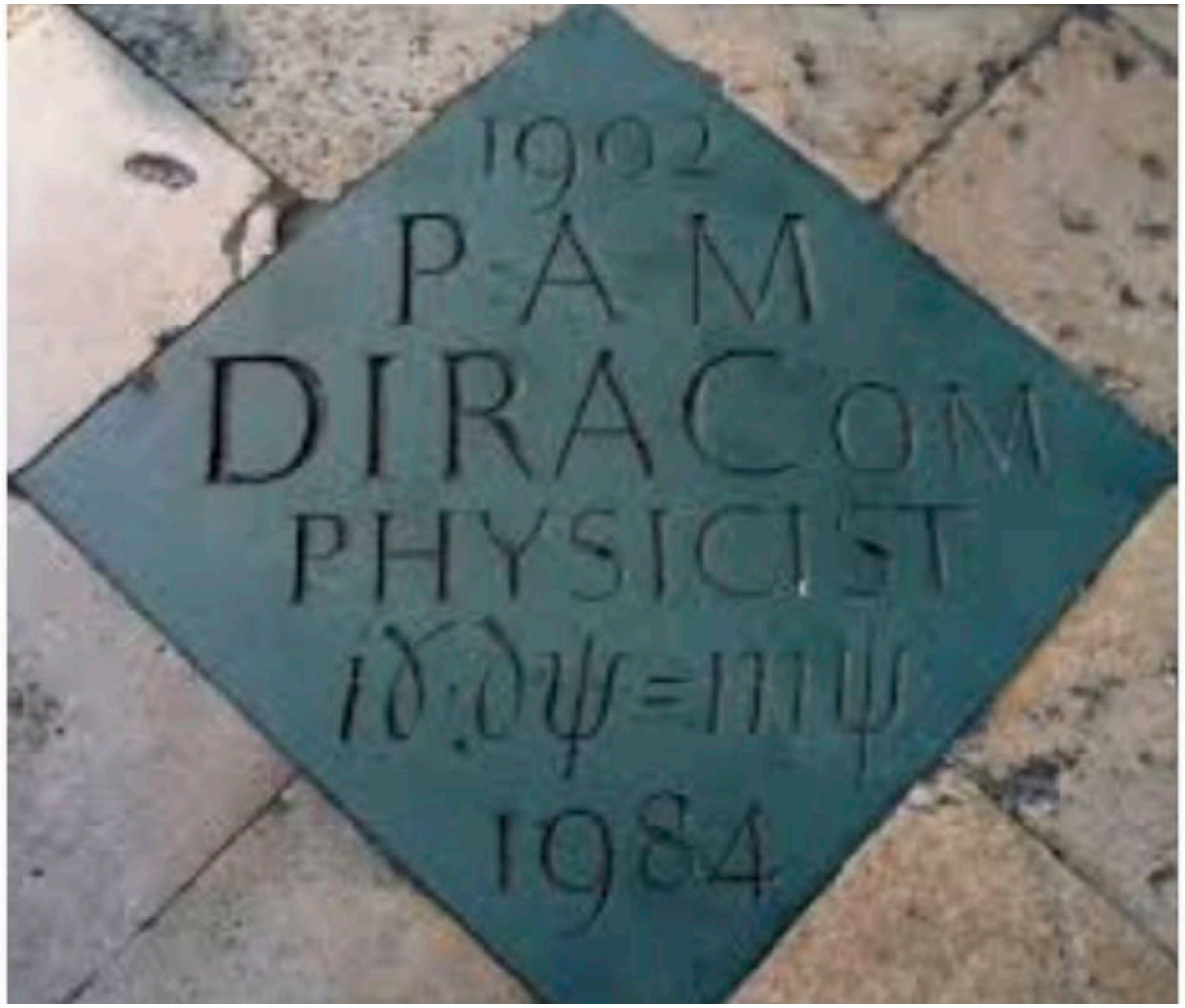
$$\vec{s} = \frac{1}{2}\hat{\sigma}$$

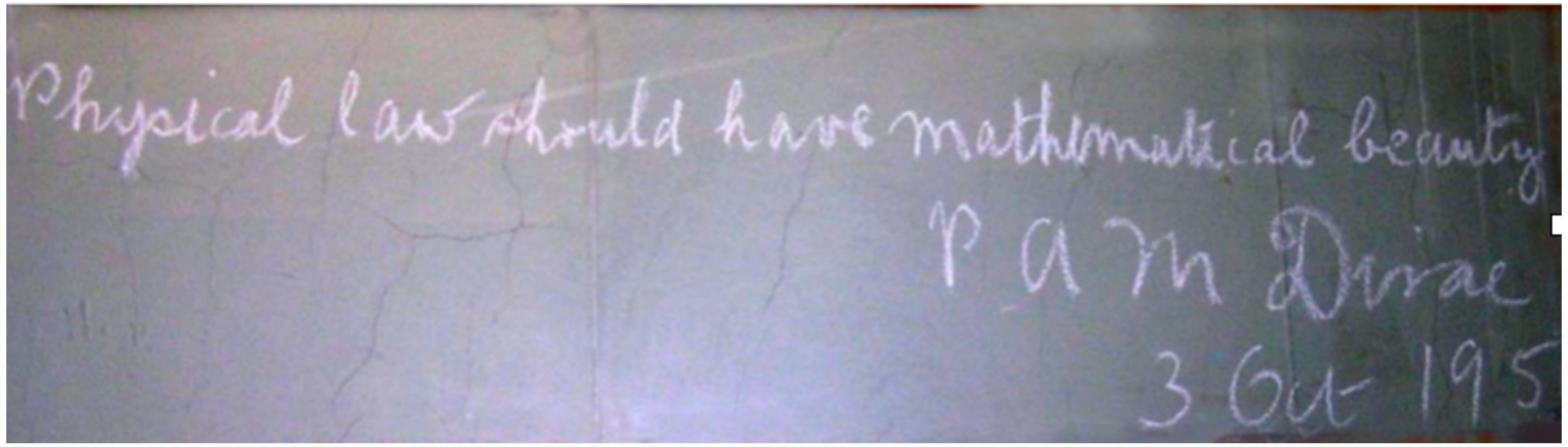
$$[s_i, s_j] = \epsilon_{ijk}s_k$$

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



English physicist
(1902-1984)





Marrying Relativity & Quantum Mechanics

$$E = \frac{p^2}{2m}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r},t)\right]\Psi(\vec{r},t)$$

$$E^2 = p^2c^2 + m^2c^4$$

$$p_\mu \equiv (p_1, p_2, p_3, p_4) = (\vec{p}, i\frac{E}{c})$$

$$\left(p_1^2 + p_2^2 + p_3^2 + p_4^2\right)^{1/2} = p_1\gamma_1 + p_2\gamma_2 + p_3\gamma_3 + p_4\gamma_4$$

$$\gamma_0^2 = 1,\ \ \gamma_1^2 = -1,\ \ \gamma_2^2 = -1,\ \ \gamma_3^2 = -1,\ \ \ \{\gamma_\mu\gamma_\nu\} = 0,\ \ \mu \neq \nu$$

Clifford algebra in 4D Minkowski space

$\vec{p} \to -i\hbar\nabla,\ E \to i\hbar\frac{\partial}{\partial t}$

We need four matrices that anti-commute:
Pauli matrices will not do the job

$$i\gamma_\mu\partial_\mu\Psi = \frac{mc}{\hbar}\Psi$$

Dirac spinner

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$i\gamma_\mu \partial_\mu \Psi = \frac{mc}{\hbar}\Psi$$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}$$

$$\left(\gamma^\mu (i\hbar\partial_\mu - eA_\mu) - mc\right)\psi = 0$$

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$(E - e\phi)\psi_+ - c\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})\,\psi_- = mc^2\psi_+$$

$$-(E - e\phi)\psi_- + c\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})\,\psi_+ = mc^2\psi_-$$

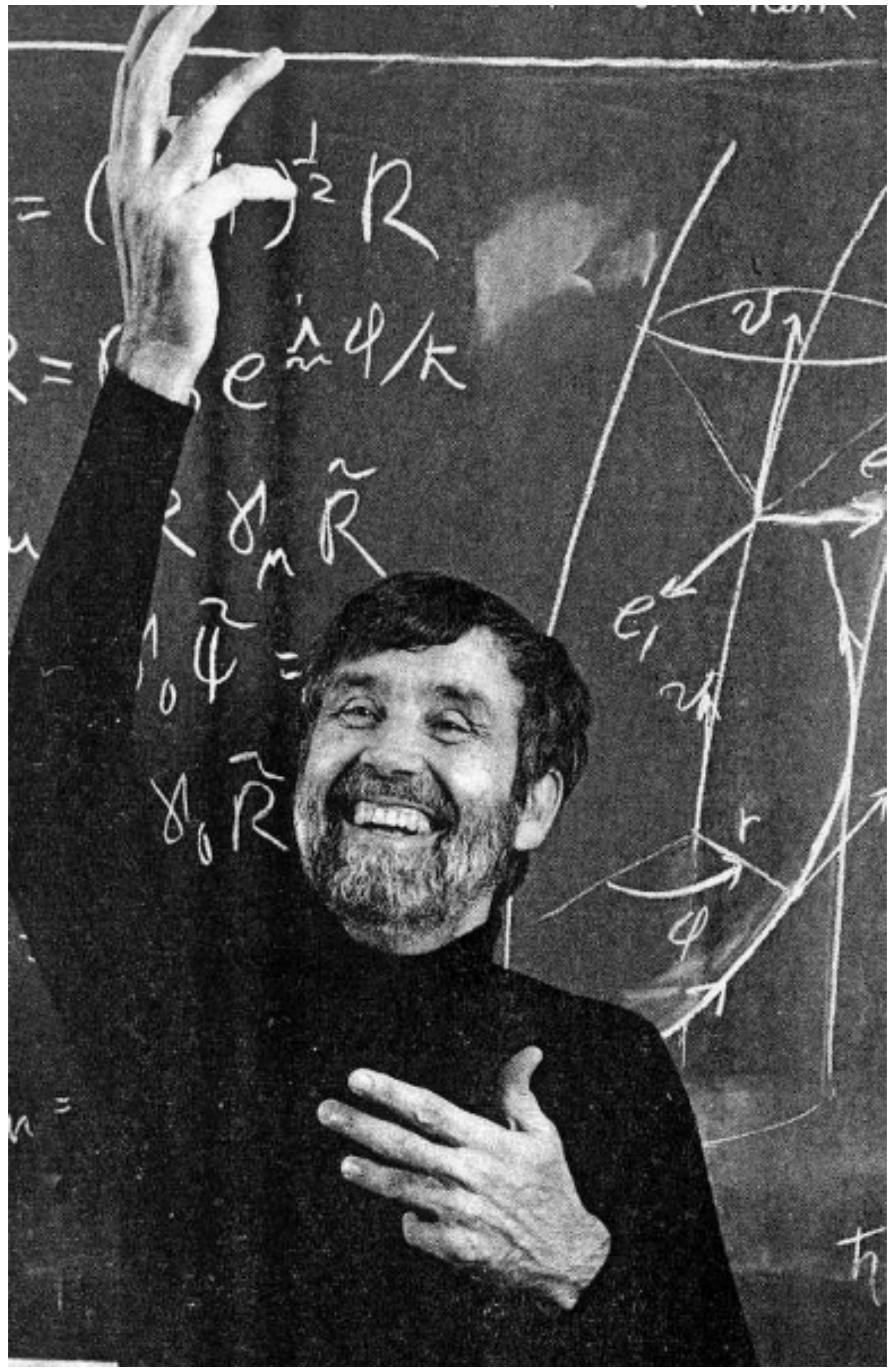
$$\psi_- \approx \frac{1}{2mc}\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})\,\psi_+$$

$$\left(E - mc^2\right)\psi_+ = \frac{1}{2m}\left[\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})\right]^2 \psi_+ + e\phi\psi_+$$

What is Spin ???

Spin emerges as intrinsic angular momentum- an abstract quantity, and is not associated with any visual picture of a spinning top. The idea that quantum spin describes some kind of *spinning* motion is *erased* in the physics literature.

David Hestenes: Spacetime Algebra & its Implication for Quantum Spin



..... It has been my privilege to pick up where Clifford left off—to serve, so to speak, as principal architect of Geometric Algebra and Calculus as a comprehensive mathematical language for physics, engineering and computer science.

Pauli, Dirac **1920s**:……

In physics, "*spin-gene*" is encoded in Pauli & Dirac matrices.

In Quantum Mechanics:

$$(x, p) \to (x, \frac{\hbar}{i}\partial_x)$$

spin as a Pauli & Dirac matrices

Hestenes **1960s**:

Pauli matrices represent a frame of three orthonormal vectors in 3D Euclidean space ; anticommutativity expresses orthogonalization.

Similarly, Dirac matrices γ_μ as orthogonal vector basis in 4D Minkowski space.

Spin is not intrinsically related to Pauli or Dirac matrices

David Hestenes's Spacetime Algebra (STA) & Deeper Dive into Quantum Spin

(1) Represent Pauli or Dirac spinors as multivectors. (Ψ- related to Rotors- unit quaternion)

(2) Pauli matrices $\hat{\sigma}_i$ or Dirac matrices $\hat{\gamma}_\mu$ are orthogonal vectors in 3D or 4D spacetime.

No more matrices and all quantities have geometric interpretation.

Spinner Ψ satisfies real equation: NO more imaginary number

$$|\psi>= \begin{bmatrix} \psi_\uparrow \\ \psi_\downarrow \end{bmatrix} \rightarrow \Psi = a_0 + \epsilon_{ijk}\hat{\sigma}_i\hat{\sigma}_j a_k = \sqrt{\rho}e^{i_g\frac{\beta}{2}}R, \quad R = \cos\frac{\theta}{2} + i_g\hat{\sigma}_i\hat{\sigma}_j\sin\frac{\theta}{2}$$

$$|\psi>= \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \rightarrow \Psi = A_0 + \hat{\gamma}_i\hat{\gamma}_0 A_k = \sqrt{\rho}e^{i_g\frac{\beta}{2}}R, \quad R = \cos\frac{\theta}{2} + i_g\hat{\gamma}_i\hat{\gamma}_j\sin\frac{\theta}{2}$$

Define SPIN:

$$S = R\hat{\gamma}_1\hat{\gamma}_2\tilde{R}$$

$$S|\psi> = \frac{\hbar}{2}i|\psi>$$

Pauli-Schödinger-Hestenes equation

$$\hbar\partial_t\Psi\hat{\sigma}_1\hat{\sigma}_2 = (-\frac{\hbar^2}{2m} + V)\Psi - \frac{e\hbar}{2mc}B\Psi\hat{\sigma}_3,$$

Dirac- Hestenes's equation

$$\hbar\Box\Psi\gamma_2\gamma_1 - \frac{e}{c}A\Psi = mc\Psi\hat{\gamma}_0,$$

NOTE: $i = \sqrt{-1}$ maps into a bivector

$$i \leftrightarrow \hat{\sigma}_2\hat{\sigma}_1, \quad i \leftrightarrow \hat{\gamma}_2\hat{\gamma}_1$$

Spinning point particle- what does it mean ?

Dirac-Hestenes spinor Ψ is a ROTOR - theory of spinning frames on the spacetime.

What physical entity is spinning as spinning frame is not a spinning thing - the spin ?

Interpreting spin as a dynamical property of electron motion.

Spin of electron describe circulation of electron mass and charge.

Electron is executing helical motion- zitterbewegung (zitter), which manifests in spin

This is a new version of de Broglie's original hypothesis that the electron has an internal clock with period precisely equal to twice the zitter period.

Coulomb field of electron is actually time average of a more basic periodic electromagnetic field oscillating with the de Broglie of frequency $\omega = \frac{mc^2}{h} \approx 10^{21} s^{-1}$

High frequency electromagnetic field (or wave) is permanently attached to electron

This gives zero-point angular momentum associated with the zero-point energy of the electron

Heisenberg uncertainty- interpreting electron spin as minimum orbital angular momentum.

Modeling the **Electron** with **Geometric Algebra**

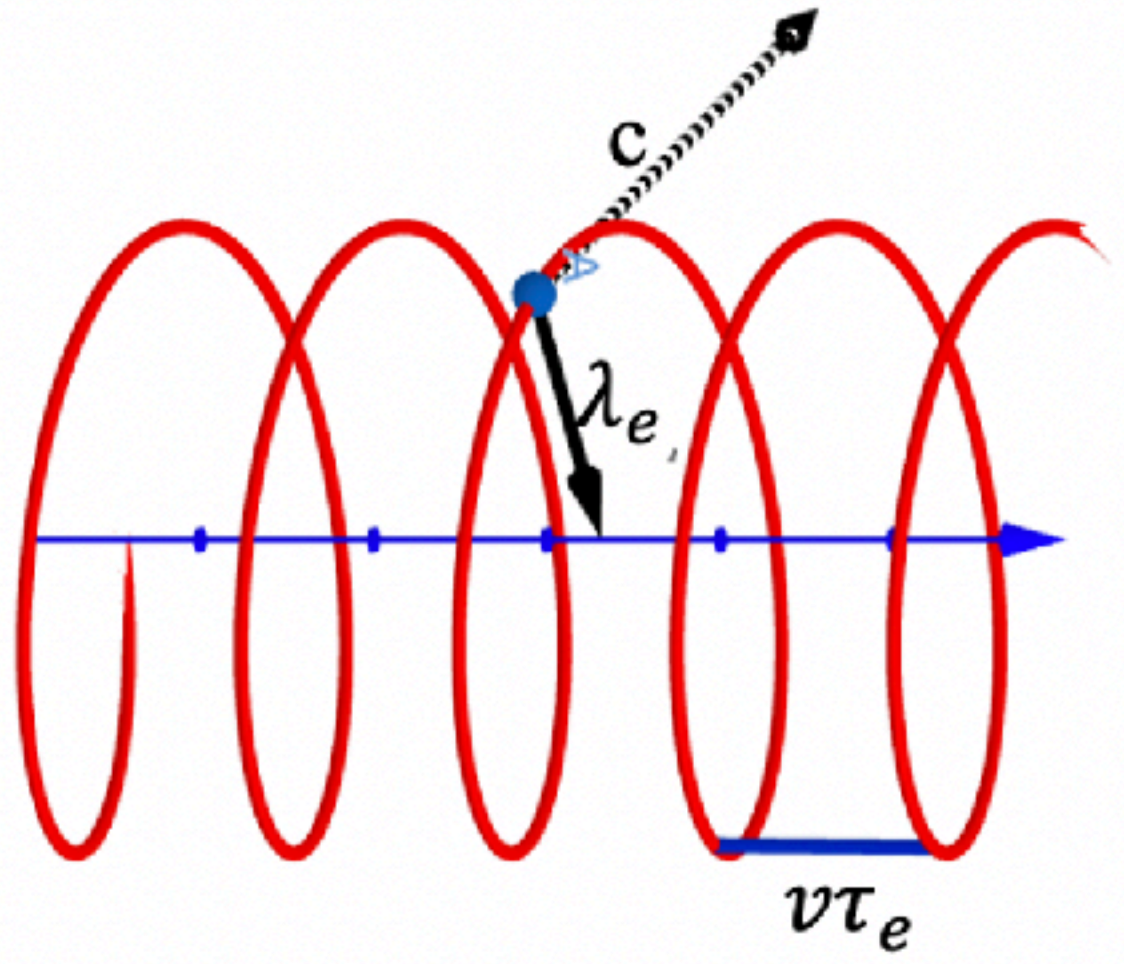
A report on work in progress

David Hestenes
Arizona State University

ICACGA 2022

"You know, it would be sufficient to really understand the electron!"
— *Einstein* (1943)

Electron **zitter** (zitterbewegung)



if the particle is considered as containing a rest energy mc^2 = hν_0, *it was natural to compare it to a small clock of frequency* ν *so that when moving with velocity* v = βc, *its frequency is different from that of the wave,* $\nu = \nu_0\sqrt{1-\beta^2}$ Louis de BROGLIE,

***"Science is the belief in the ignorance of experts"* Feynman**