SSQII JMM2023

Coquaternion as a functional module of a biologic system

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Coquaternions c**H**:= w**1** +x**i** +y**j** + z**k** , imaginary basis elements as feedback patterns of a biologic system

Coquaternions are four-element structures over **R** with {1,x,y,z} basis elements. They fill 4D vector space over real numbers.

Table of multiplication of basis elements and cH conjugates w1-xi-yj-zk show c**H** as **closed** algebraic structure under multiplication of its elements

ij= k, ji=-k, ki=j, ik=-j, kj=i, jk=-i, ii=-1, jj=1, kk=1, ijk=1

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	1	-i
k	k	j	i	1

Negative feedback **i** $\approxeq S_0 = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$

Positive feedback **j** $\approxeq S_2 = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}$

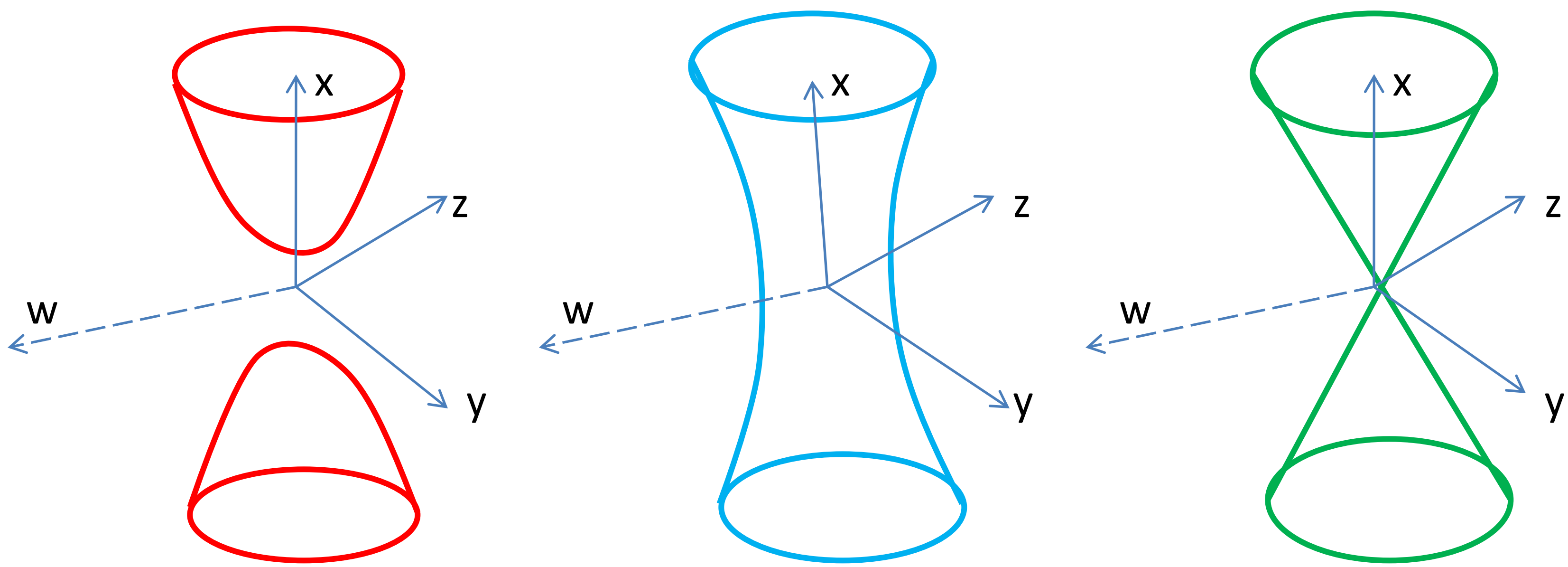
Reciprocal links **k** $\approxeq S_1 = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}$

Arbitrary 2x2 matrix over real numbers is isomorphic to c**H**

Geometric images of coquaternions (cH) related to its isotropic quadratic form

$cH = w1 + xi + yj + zk$ $<cH,cH> = w^2 + x^2 - y^2 - z^2$

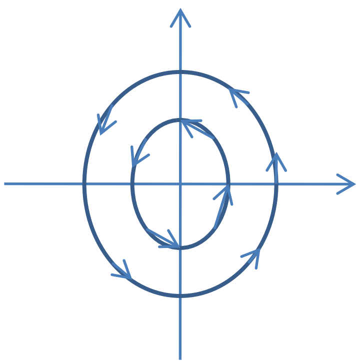
$w^2 + x^2 - y^2 - z^2 < 0$ $w^2 + x^2 - y^2 - z^2 > 0$ $w^2 + x^2 - y^2 - z^2 = 0$



Three surfaces (two-sheet hyperboloid, one-sheet hyperboloid and double-cone) of equipotential energy levels. **w** is orthogonal to 3D semi-Euclid space (**x**, **y**, **z**)

Sketch of dynamical images of NFB, PFB and RL patterns obtained from ODE.

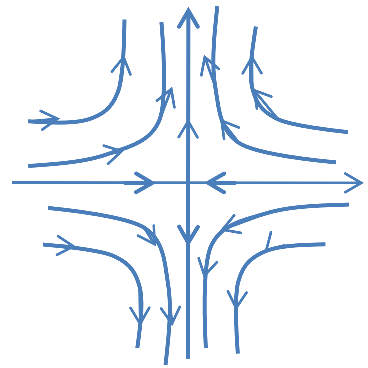
Negative feedback (NFB)



$$S_0 = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$$

d**v**/dt=S_0**v**

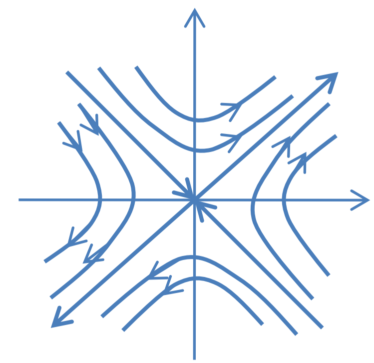
Reciprocal links (RL)



$$S_1 = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}$$

d**u**/dt=S_1**u**

Positive feedback (PFB)

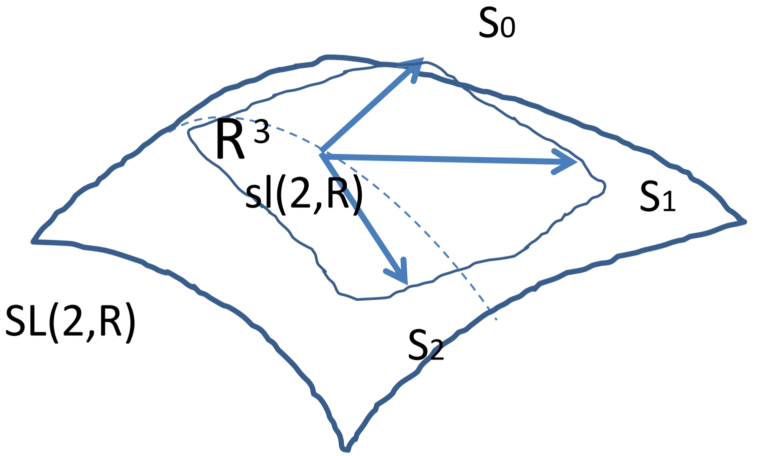


$$S_2 = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}$$

d**w**/dt=S_2**w**

Matrices of NFB, PFB and RL are operators of ODE. Variables are expressed in a vector form.
< **S_0, S_1, S_2**> represent basis elements of **Lie algebra sl(2,R)** and **coquaternions.**
{**S_0, S_1, S_2** } are **tracelees** matrices determining non expanding, same-energy level processes.

NFB, PFB and RL (PNR) matrices as basis elements of Lie algebra sl(2,R) of a special linear group form 3D linear space over real numbers



sl(2,R) is an **additive** group: $S = aS_0 + bS_1 + cS_2$

Lie bracket: $[S_i, S_j] = S_iS_j - S_jS_i \neq 0$

Physiologic importance to be an additive group determines **integrative** properties of functional elements including PNR basis.

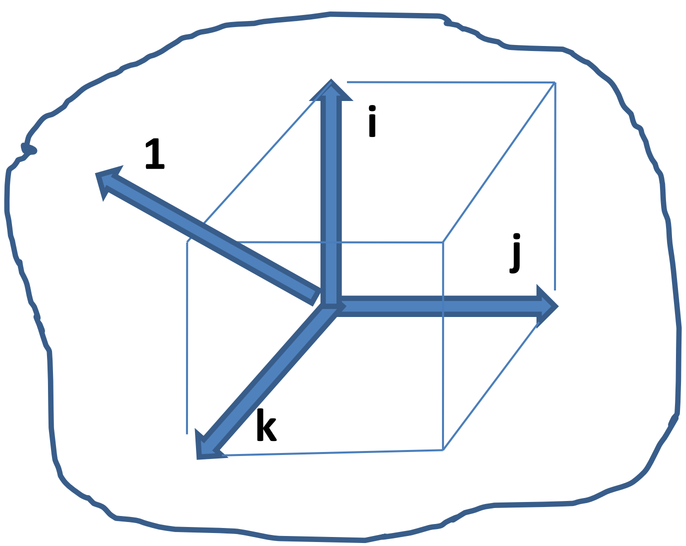
NFB, PFB and RL matrices as basis elements of sl(2,R). Elements of sl(2,R) lie on the tangent to the group SL(2,R) space. The algebra has **non-commutative** property. It is closed as a group under addition of its elements.

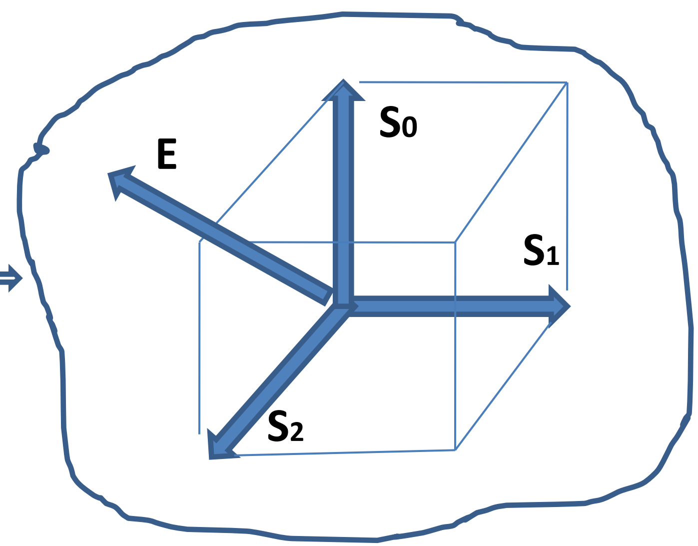
Integration of basis patterns (PNR) in a whole functional structure of a biologic system.

Coquaternion as a 4D vector space

4D functional structure of a 2D biologic module (system)

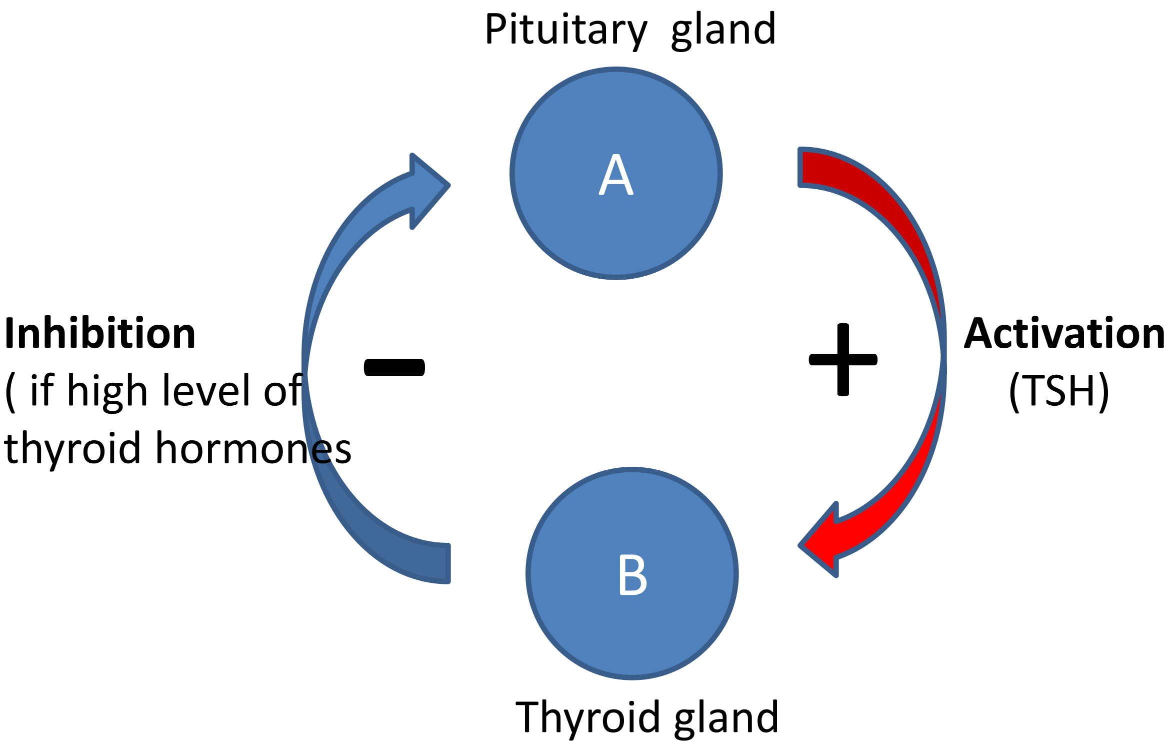
$c\mathbf{H} := w1 + xi + yj + zk \quad \longleftrightarrow \quad BS := w\mathbf{E} + x\mathbf{S_0} + y\mathbf{S_1} + z\mathbf{S_2}$



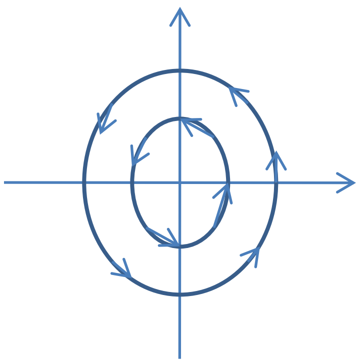


A functional biologic **module** is a morphofunctional unit regulated by coquaternion structure

Negative Feedback



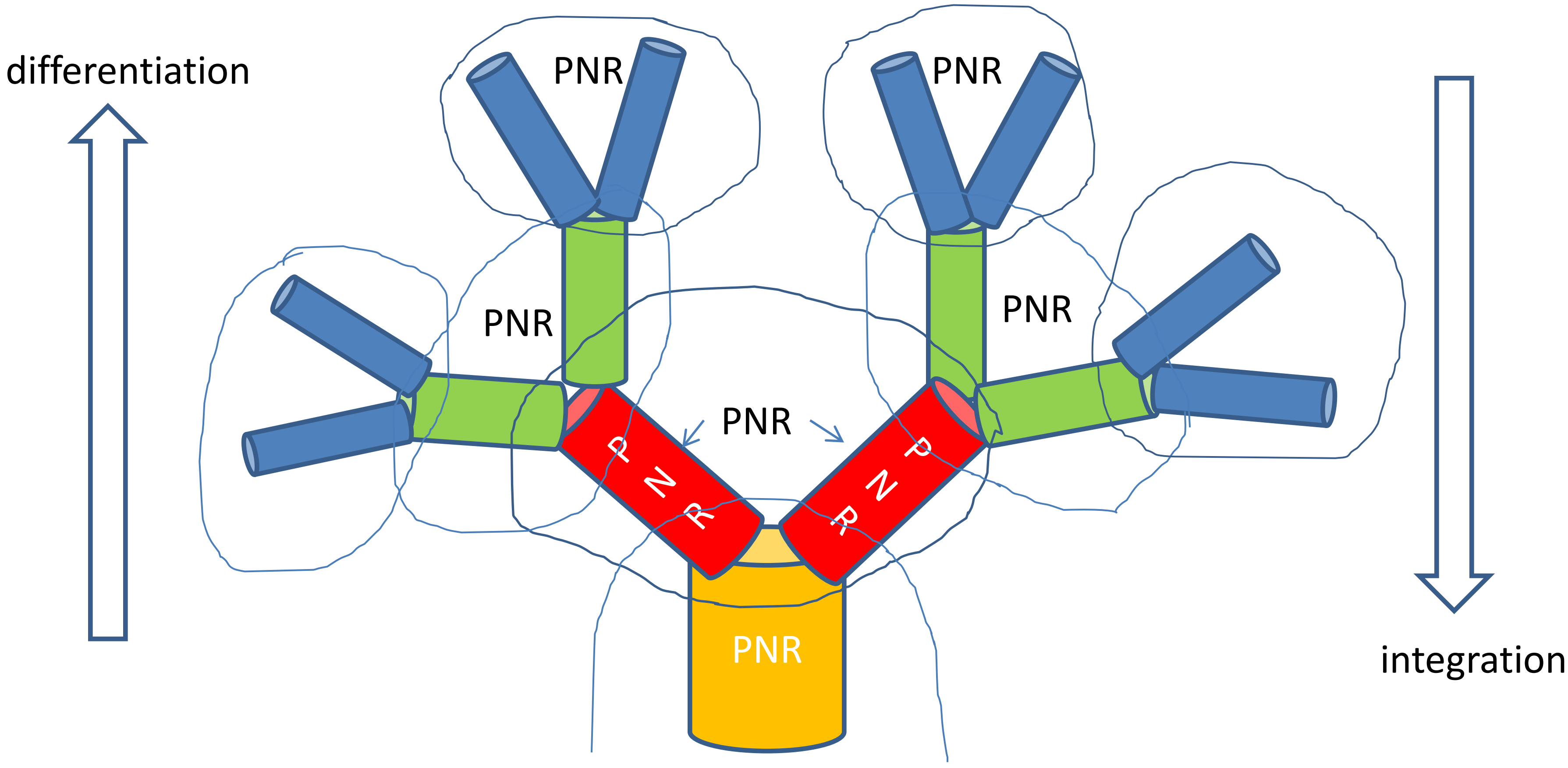
Negative feedback (NFB)



$$S_0 = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$$

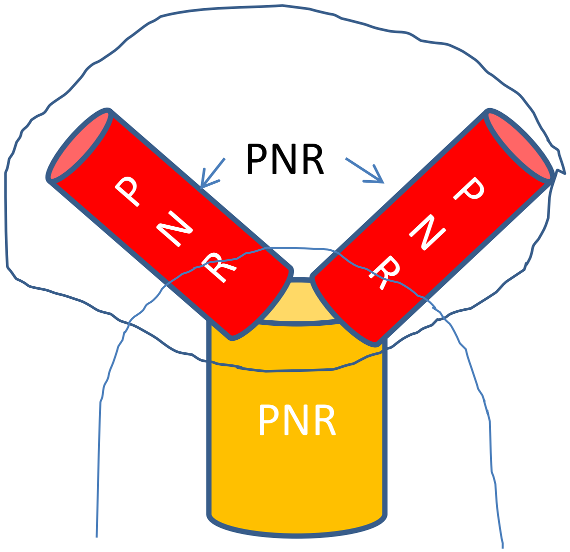
NFB is a regulatory pattern based on the principle of inhibition of subsystem (A) when activation of another subsystem (B) exceeds certain level. Subsystem A will begin receiving inhibitory stimuli in order to diminish the level of activation. The result of this interactions will keep the system within certain functional margins.

Hierarchical organization of biologic systems and phylo-ontogenetic tree.

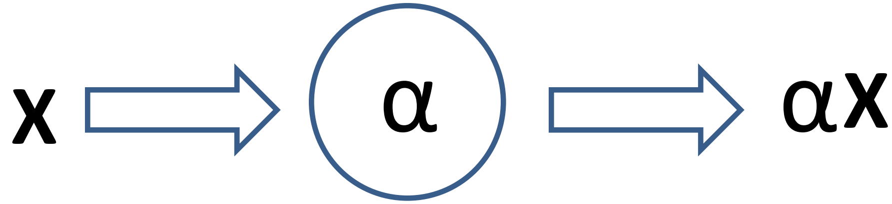


Each branch is a self-regulating BS. Same-colored branches represent differentiated BS linked by PNR patterns. PNR= Positive feedback; Negative feedback; Reciprocal links

Fragment of phylo,-ontogenetic tree showing splitting of PNR base elements between two differentiated subsystems (red)



Second order operator A represents internal functional structure of a biologic system



Scalar 1D operator **α** represents undifferentiated structure of a biologic system.
It transforms input **x** into output **αx**

$$A = \left\{ \begin{pmatrix} \mu & \xi \\ \nu & o \end{pmatrix} : (\mu, \nu, \xi, o) \in \mathbb{R} \right\}$$

$$\lambda_{1,2} = 1/2\left[(\mu + o) \pm \sqrt{(\mu - o)^2 + 4\xi\nu}\right]$$

Matrix **A** of second order operator represents functional structure of a 2-element system. Diagonal form has two eigenvalues $\lambda_{1,2}$ as characteristics of a functional structure of two reciprocally linked differentiated subsystems. Eigenvalues act on 1D eigenvectors representing split morphofunctional elements (subsystems).

Matrices M(2,R) are isomorphic to coquaternions

$$A = \left\{ \begin{pmatrix} \mu & \xi \\ \nu & o \end{pmatrix} : (\mu, \nu, \xi, o) \in \mathbb{R} \right\}$$

$$A \leftrightarrow q = \tfrac{1}{2}\left[(\mu + o)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\xi - \nu)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + (\nu + \xi)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (\mu - o)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right].$$

$$A \cong q = \begin{pmatrix} a + d & b + c \\ b - c & a - d \end{pmatrix}.$$

$$\lambda_{1,2} = a \pm \sqrt{b^2 + d^2 - c^2}$$

Matrices M(2,R) are isomorphic to the set of cH. {a, b, c, d} are linear combinations of initial matrix coefficients. Matrices A written in equivalent forms. Eigenvalues of diagonal form can't be presented as linear combinations of basis matrices.

Linear combinations of 4x4 complex matrices represent functional conditions and hierarchical structure of the system consisting of two differentiated subsystems after splitting

$$\pm\sqrt{b^2 + d^2 - c^2} = \alpha b + \beta d + \gamma c$$

$$\alpha\beta = -\beta\alpha, \alpha\gamma = -\gamma\alpha, \qquad \beta\gamma = -\gamma\beta$$

$$\alpha = i\begin{pmatrix} 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \end{pmatrix} \beta = i\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \gamma = i\begin{pmatrix} 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \end{pmatrix}$$

In order to linearize expression containing the square root on the left side of the equation **α**, **β** and **γ** quantities on the right side are presented as 4x4 complex matrices. Block-diagonal form shows isomorphism with imaginary basis elements of coquaternion