Geometry and Algebra in the Hopf Fibration

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Outline

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The Hopf fibration (1931)

$S^3 \to S^2$

$$(a, b, c, d) \to \big(2(ac + bd), 2(bc - ad), a^2 + b^2 - c^2 - d^2\big)$$

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- applications to magnetic monopoles, rigid body mechanics, quantum information theory
- has lovely geometry and algebra

Fibers of the Hopf map

Exercise (Bröcker, tom Dieck)

Show that any two fibers of the Hopf fibration are linked in S^3.

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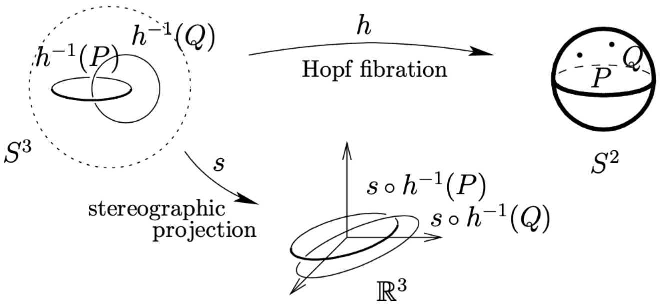

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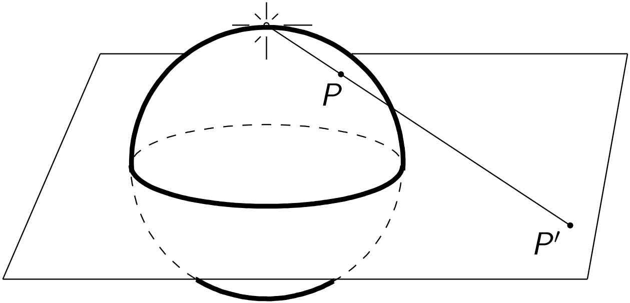
Stereographic projection $S^2 \to \mathbb{C} \cup \{\infty\}$



$$P = (a, b, c) \to \frac{a + ib}{1 - c} = P'$$

Original version of the Hopf map

"Classic Hopf"

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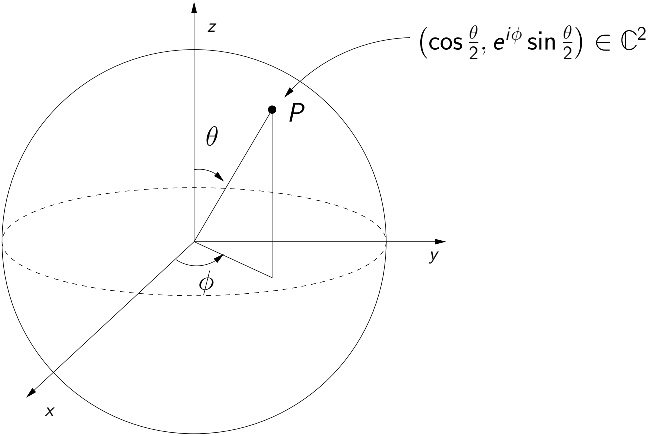
Bloch sphere, cont'd



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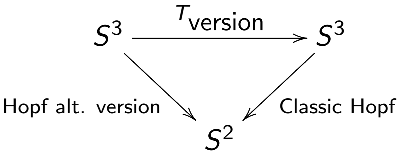
Reconciling the versions



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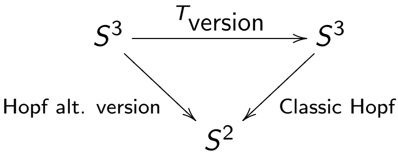

version	T_{version}
Quaternion Hopf	$(a, b, c, d) \to (a, d, c, b)$
Möbius Hopf	$(a, b, c, d) \to (a, b, -c, d)$
Bloch Hopf	$(a, b, c, d) \to (a, -b, c, -d)$

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More algebra and geometry

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Rotations
$SU(2)$, unit quaternions, Möbius elliptical group

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$SU(2)$, maximal torus T, homogeneous space $SU(2)/T$

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two more versions of the Hopf fibration

$$\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}(\mathbb{C}^2)$$
$$SU(2) \to SU(2)/T$$

Comparison of rotation conventions

Ways that $(a, b, c, d) \in S^3$ acts as a rotation on S^2

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\begin{array}{lll}
\text{(unit quaternions)} & \leftrightarrow \text{(Möbius elliptic group)} & \leftrightarrow SU(2) \\
a + bi + cj + dk & \leftrightarrow \left[z \to \frac{(a+bi)z+(c+di)}{(-c+di)z+(a+bi)} \right] & \leftrightarrow \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}
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	axis	angle
quaternions	(b, c, d)	$2\cos^{-1} a$
Möbius transf.	$(d, -c, b)$	$2\cos^{-1} a$
Bloch coordinates	(d, c, b)	$-2\cos^{-1} a$

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- spherical (elliptic Möbius) geometry
- group actions
- Lie groups, homogeneous spaces
- Bloch coordinates for 1-qubit states

Thank you!

http://quantum.lvc.edu/mathphys

References

[1] Heinz Hopf.
Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche.
Mathematische Annalen, 104:693–665, 1931.

[2] Theodor Bröcker and Tammo tom Dieck.
Representations of Compact Lie Groups.
Springer, 1985.

More References

[1] David W. Lyons.
An elementary introduction to the Hopf fibration.
Mathematics Magazine, 76(2):87–98, 2003.

[2] David W. Lyons.
Survey of Hopf fibrations and rotation conventions in mathematics and physics.
arXiv:0808.3089v2 [math-ph], September 2008.

[3] David W. Lyons.
Introduction to Groups and Geometries.
January 2021 edition, 2020.
https://mathvista.org.