Quaternions and Reproducing kernel spaces

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Outline of the talk:

- Positive definite functions and examples.
- Reproducing kernel Hilbert spaces and reproducing kernel Pontryagin spaces (Krein spaces will also appear).
- Some important examples (complex setting). Bargmann-Fock-Segal space, Drury-Arveson space, de Branges Rovnyak space.
- Why one can consider the quaternionic setting. Different kinds of analyticity.
- Some examples from the quaternionic setting (hypercomplex and Fueter series; slice hyperholomorphic).

Important cases not touched upon in this talk:

Grassmann algebra, Clifford algebra and poly-slice analytic functions, ternary algebras, bicomplex numbers and others.

Two highlights:

- The spectral theorem holds for Hermitian quaternionic matrices: if $M \in \mathbb{H}^{n \times n}$ and such that $H = H^*$,

$$M = UDU^*$$

  with D diagonal with real entries and U unitary
- One has (Fueter, 1936) three non-commuting variables which are analytic.

Positive definite functions:

The function (kernel) $k(z,w)$ is positive definite on Ω (*misleading but accepted terminology*) if for all choices of $N\in\mathbb{N}$, $z_1,\ldots,z_N\in\Omega$ and $c_1,\ldots,c_N\in\mathbb{C}$ we have:

$$\sum_{i,j=1}^{N}\overline{c_i}k(z_i,z_j)c_j\geq 0.$$

For every choice of $N\in\mathbb{N}$ and of points $z_1,\ldots,z_N\in\Omega$ the matrix

$$\begin{pmatrix} k(z_1,z_1) & k(z_1,z_2) & \cdots & k(z_1,z_N)\\ k(z_2,z_1) & k(z_2,z_2) & \cdots & k(z_2,z_N)\\ \vdots & \vdots & & \vdots\\ \vdots & \vdots & & \vdots\\ k(z_N,z_1) & k(z_N,z_2) & \cdots & k(z_N,z_N)\end{pmatrix}\geq 0.$$

Negative squares:

One now asks that all the above matrices has at most $\kappa<\infty$ strictly negative eigenvalues (same κ for *all* matrices).

Example: Bochner's theorem

Let $\Omega = \mathbb{R}$ and $K(x,y) = F(x-y)$ be a complex-valued continuous function which satisfies

$$\sum_{i,j=0}^{N} F(x_i - x_j) c_i \bar{c}_j \geq 0 \quad \forall N \quad \forall c_i \in \mathbb{C} \quad \forall x_1, x_2, ..., x_n \in \mathbb{R}$$

Then there exists a positive measure μ such that

$$F(u) = \int_{\mathbb{R}} e^{-ixu} d\mu(x)$$

$F(u)$ is the Fourier transform of a positive measure.

$$\int_{\mathbb{R}} \frac{e^{itu}}{1+u^2} du = \pi e^{-|t|}, \quad t \in \mathbb{R},$$
$$\int_{\mathbb{R}} e^{itu} e^{-|u|} du = \frac{2}{1+t^2}$$

Examples

The functions

$$e^{-|t-s|} \quad \text{and} \quad \frac{1}{1+(t-s)^2}$$

are positive definite on the real line.

Gaussian:

$$\int_{\mathbb{R}} e^{-\frac{u^2}{2}} e^{-iut} du = \sqrt{2\pi} e^{-\frac{t^2}{2}}.$$

The function $e^{-\frac{(t-s)^2}{2}}$ is positive definite on the real line. Can also see it via

$$e^{-\frac{(t-s)^2}{2}} = e^{-\frac{t^2}{2}} e^{ts} e^{-\frac{s^2}{2}}$$

Reproducing kernel Hilbert spaces.

Definition: A reproducing kernel Hilbert space $\mathcal{H}$ is a Hilbert space of **functions** on a set (say Ω), for which the point evaluations are bounded.

By Riesz theorem, $\exists k_z \in \mathcal{H}$ (notation: $k(\cdot, z)$), called the **reproducing kernel**, such that

$$f(z) = \langle f, k_z \rangle_{\mathcal{H}}, \quad \forall f \in \mathcal{H}.$$

RK and positive definite functions

There is a one-to-one correspondence between positive definite functions and reproducing kernel Hilbert spaces.

There is a one-to-one correspondence between functions with a finite number of negative squares and reproducing kernel Pontryagin spaces.

There is a onto (but not one-to-one) correspondence between difference of positive definite functions and reproducing kernel Krein spaces (Laurent Schwartz, 1964).

Definition:

The function $k(z, w)$ defined for $z, w \in \Omega$ is factorizable if there is a pre-Hilbert space $\mathcal{H}$ and a $\mathcal{H}$-valued function $z \mapsto h_z$ such that

$$k(z, w) = \langle h_w, h_z \rangle_{\mathcal{H}}$$

Theorem:

A function is factorizable if and only if it is positive definite.

Link with machine learning and support vector machines:

Usually $z \in \mathbb{R}^n$ is called the feature vector and h_z is called the feature map and belongs to a larger space, which allows separation of data. One works with the kernel $k(z, w)$ and not with h_z to make the computations.

Stochastic processes.

There is a one-to-one correspondence between covariance of Gaussian centered processes and positive definite functions.

Example.

Not always easy to recognize if a function is positive definite. Is $|t|^{2H}+|s|^{2H}-|t-s|^{2H}$ ($H\in(0,1)$) positive definite on the real line?

$$|t|^{2H}+|s|^{2H}-|t-s|^{2H}=$$

$$=c_H\int_0^\infty \frac{(1-\cos(tu))(1-\cos(su))+\sin(tu)\sin(su)}{u^2}\frac{du}{u^{2H-1}},$$

where $c_H>0$ depends only on H.

It is the covariance function of the fractional Brownian motion (H is the Hurst parameter; Brownian motion for $H=1/2$).

Remarks:

Not every Hilbert space of functions is a reproducing kernel Hilbert space.

Counterexamples involve the axiom of choice and the fact that a vector space basis in an infinite dimensional Hilbert space is uncountable.

Donoghue and Masani (1983) *A class of invalid assertions concerning function Hilbert spaces*:

Given an infinite dimensional Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and $x_0 \neq 0 \in \mathcal{H}$. The map $x \mapsto \langle x, x_0\rangle_{\mathcal{H}}$ is continuous with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, but:

There exists an inner product $(\cdot,\cdot)$ on $\mathcal{H}$ for which $\mathcal{H}$ is still a Hilbert space but the map $x \mapsto \langle x, x_0\rangle_{\mathcal{H}}$ is not continuous with respect to $(\cdot,\cdot)$.

The Bargmann-Fock-Segal space $\mathcal{F}$:

The space of entire functions f such that ($s = x + iy \in \mathbb{C}$)

$$\langle f, f\rangle_{\mathcal{F}} = \frac{1}{\pi}\iint_{\mathbb{C}} |f(s)|^2 e^{-|s|^2} dxdy < \infty$$

Then, $k_z \ : \ s \mapsto e^{s\overline{z}} \in \mathcal{F}$ and one has the reproducing kernel property:.

$$\langle f, k_z\rangle_{\mathcal{F}} = f(z), \quad \forall z \in \mathbb{C}$$

Reproducing kernel:

Point evaluations are bounded.

$z \mapsto e^{z\overline{w}} \in \mathcal{F}$ is the reproducing kernel.

Analytic characterization:

$$\langle z^n, z^m \rangle = \frac{1}{\pi} \iint_{\mathbb{C}} z^n \overline{z^m} e^{-|z|^2} dxdy = n! \delta_{n,m}, \quad n, m \in \mathbb{N}_0,$$

with $\delta_{n,m}$ being the Kronecker symbol. So:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F} \iff \sum_{n=0}^{\infty} n! |a_n|^2 < \infty.$$

Property:

The Fock space can be characterized as the unique space of entire functions where the adjoint of differentiation is multiplication by z.

$$\partial^* = M_z \quad \text{and} \quad M_z^* = \partial.$$

Motivation from quantum mechanics. Work of Fock in 1928.

Bargmann transform:

The map

$$B(f)(z) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} f(u) e^{-\frac{u^2}{2} + \sqrt{2} z u} du$$

is unitary from $\mathrm{L}^2(\mathbb{R}, du)$ onto the Fock space

Properties:

In the Fock space, annihilation operator is ∂ and creation operator is M_z. With $F(z) = B(f)(z)$, one has:

$$(B(uf(u)))(z) = \frac{F'(z) + zF(z)}{\sqrt{2}} \quad \text{(image of the position operator)}$$

$$(B(f'(u)))(z) = \frac{F'(z) - zF(z)}{\sqrt{2}} \quad \text{(image of the momentum operator)}$$

Summary (and roadmap for study of other spaces):

1. We have a Hilbert space of functions.
2. Point evaluations are bounded.
3. Characterized by a function of two variables, $k(z, w) = e^{z\overline{w}}$.
4. It has geometric and analytic characterizations.
5. It has an important motivation (here, quantum mechanics; has also applications in signal processing).
6. It can be characterized in terms of a transform.

Other important examples (just name dropping ...):

Arveson space, de Branges-Rovnyak spaces, Hardy space and applications to signal processing and the theory of linear systems.

The skew–field of quaternions $\mathbb{H}$:

elements of the form

$$x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where the x_i are real and where e_0, e_1, e_2, e_3 satisfy the rules of multiplication from the Cayley table:

$\nearrow$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

Split-quaternions:

k_1, k_2 and i the basis of the split-quaternions, with multiplication table

$\nearrow$	1	k_1	k_2	i
1	1	k_1	k_2	i
k_1	k_1	1	i	k_2
k_2	k_2	$-i$	1	$-k_1$
i	i	$-k_2$	k_1	-1

Construction:

To build $\mathbb{H}$, take

$$\mathrm{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{e}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathrm{e}_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{e}_3 = \mathrm{e}_1\mathrm{e}_2.$$

$$\mathbb{H} = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}, \quad z_1, z_2 \in \mathbb{C} \right\}$$

Split quaternions:

$$\mathbb{H}_{\mathbb{R}} = \left\{ \begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}, \quad z_1, z_2 \in \mathbb{C} \right\}$$

To go to hypercomplex settings:

Need the notion of positivity. In the quaternionic setting easy. Also the spectral theorem for Hermitian matrices hold, and so can define negative squares. Can also use the map

$$\chi(A + \mathrm{e}_3 B) = \begin{pmatrix} A & \overline{B} \\ -B & \overline{A} \end{pmatrix}.$$

Remark:

In the split-quaternionic setting, consider the map

$$\psi(A + \mathrm{k}_1 B) = \begin{pmatrix} A & \overline{B} \\ B & \overline{A} \end{pmatrix}$$

Two natural adjoints there, namely corresponding to $\psi(M)^*$ and $J_0\psi(M)^*J_0$, $J_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ (indefinite inner product on the coefficient space). The second one will lead to reproducing kernel Krein spaces.

Recall that an analytic function is a complex–valued function solution of $\partial f = 0$ with $\partial = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. Furthermore, with $\overline{\partial} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ we have: $\partial\overline{\partial} = \overline{\partial}\partial = \Delta_{\mathbb{R}^2}$.

Hyper–analytic functions:

Left hyper–analytic (or hyper–holomorphic) functions are quaternionic–valued functions solutions of $Df = 0$ where

$$D = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}.$$

(we will say hyper–analytic rather than left hyper–analytic).

With

$$\overline{D} = \frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}$$

we have the factorization of the Laplacian:

$$D\overline{D} = \overline{D}D = \Delta_{\mathbb{R}^4}.$$

We identify $\mathbb{H} = \mathbb{C}^2$ via $e_1 = i$ and

$$x = z_1 + z_2 e_2$$

with $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$. Note that for $z \in \mathbb{C}$,

$$z e_2 = e_2 \overline{z}.$$

Write $f(x) = f_1(z_1, z_2) + e_2 f_2(z_1, z_2)$.
Hyper–analyticity is equivalent to the Cauchy–Riemann type equations:

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial f_2}{\partial \overline{z_2}},$$

$$\frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1}.$$

1. The product of two hyper–analytic functions need not be hyper–analytic.
2. The quaternionic variable

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

is not hyper–analytic (it is slice hyper-analytic).

Fueter variables:

The functions $\zeta_\ell(x) = x_j - e_\ell x_0$ $\ell = 1, 2, 3$ are hyper–analytic and called the Fueter (or hyper–analytic) variables.

They were introduced by Fueter (1936). They do not commute and do not play the role of independent variables, but z_ℓ "corresponds" to ζ_ℓ in some sense.

Gleason's problem for hyper–analytic functions (after A-Shapiro-Volok, JFA 2005):

$$f(x) - f(0) = \sum_{\ell=1}^{3}(x_\ell - x_0 e_\ell)\int_0^1 \frac{\partial}{\partial x_\ell} f(tx)dt.$$

The chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t} f(tx) = \sum_{\ell=0}^{3} x_\ell \frac{\partial f}{\partial x_\ell}(tx).$$

Since the function is left hyper–holomorphic,

$$\frac{\partial f}{\partial x_0} = -e_1 \frac{\partial}{\partial x_1} f - e_2 \frac{\partial}{\partial x_2} f - e_3 \frac{\partial}{\partial x_3} f.$$

Replacing $\frac{\partial f}{\partial x_0}$ by this expression we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\ t} f(tx) = \sum_{\ell=1}^{3}(x_\ell - x_0 e_\ell)\frac{\partial f}{\partial x_\ell}(tx).$$

Fueter series:

Iterating the formula

$$f(x) - f(0) = \sum_{\ell=1}^{3} (x_\ell - x_0 e_\ell) \int_0^1 \frac{\partial}{\partial x_\ell} f(tx) dt$$

one obtains the development of a left hyper–analytic function in a series of Fueter polynomials.

$$f(x) = \sum_{\nu \in \mathbb{N}_0{}^3} \zeta^\nu(x) f_\nu, \qquad f_\nu \in \mathbb{H},$$

where $\zeta^\nu(x) = \zeta_1(x)^{\times \nu_1} \times \zeta_2(x)^{\times \nu_2} \times \zeta_3(x)^{\times \nu_3}$ and the symmetrized product of $a_1, \ldots, a_n \in \mathbb{H}$ is defined by

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}$$

where S_n is the set of all permutations of the set $\{1, \ldots, n\}$.

Same method works for split quaternions and ternary algebras. General approach in A-Paiva-Struppa, Israel Journal Math, 2020.

$z^n = z_1^{n_1} z_2^{n_2} \cdots$ in SCV is replaced by $\zeta^{\times\alpha}$.

Definition (F. Sommen, 1988):

The Cauchy–Kovalevskaya product $f \odot g$ of the hyper–analytic functions

$$f = \sum_{\nu \in {\mathbb{N}_0}^3} \zeta^{\nu} f_{\nu}, \; g = \sum_{\nu \in {\mathbb{N}_0}^3} \zeta^{\nu} g_{\nu}$$

is defined by (convolution on the coefficients)

$$f \odot g = \sum_{\eta \in {\mathbb{N}_0}^3} \zeta^{\eta} \sum_{0 \leq \nu \leq \eta} f_{\nu} g_{\eta-\nu}.$$

It is by construction hyper–analytic at the origin.

Why is it called Cauchy–Kovalevskaya product? Other definition in terms of the Cauchy–Kovalevskaya extension theorem from PDEs.

Theorem (the quaternionic Arveson space):

The function

$$k_y(x) = (1 - \zeta_1(x)\overline{\zeta_1(y)} - \zeta_2(x)\overline{\zeta_2(y)} - \zeta_3(x)\overline{\zeta_3(y)})^{-\odot}$$
$$= \sum_{\nu\in\mathbb{N}_0{}^3} \frac{|\nu|!}{\nu!}\zeta^{\nu}(x)\overline{\zeta^{\nu}(y)}$$

is positive definite in the ellipsoid $\Omega = \left\{x \in \mathbb{H} \mid 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1\right\}$. The reproducing kernel right-Hilbert space $\mathrm{H}(k)$ with reproducing kernel $k_y(x)$ is the set of functions of the form

$$f(x) = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \zeta^{\nu}(x) f_{\nu},$$

endowed with the $\mathbb{H}$-valued inner product

$$\langle f, g\rangle = \sum_{\nu\in\mathbb{N}_0{}^3} \frac{\nu!}{|\nu|!}\overline{g_{\nu}} f_{\nu}.$$

Identity:

Let

$\mathcal{M}_{\zeta_n} f = \zeta_n \odot f, \ \ n = 1, 2, 3$

and let

$\mathcal{C} : \mathrm{H}(k) \mapsto \mathbb{H}$ be the operator of evaluation at the origin: $\mathcal{C}f = f(0)$.

Then,

$$I - \mathcal{M}_{\zeta_1}\mathcal{M}_{\zeta_1}^* - \mathcal{M}_{\zeta_2}\mathcal{M}_{\zeta_2}^* - \mathcal{M}_{\zeta_3}\mathcal{M}_{\zeta_3}^* = \mathcal{C}^*\mathcal{C}$$

and, with $\mathcal{R}_n f(x) = \int_0^1 \frac{\partial}{\partial x_\ell} f(tx)dt$,

$$\mathcal{M}_{\zeta_j}^* = \mathcal{R}_j.$$

de Branges Rovnyak spaces:

Function $s(x)$, left hyperholomorphic in Ω, is a Schur function if the C-K multiplication operator

$$M_s f = s \odot f$$

is a contraction from $\mathcal{H}(k)$ into itself.
Hence $s(x)$ is a Schur function iff the kernel

$$\begin{aligned} k_s(x,y) &= ((I - M_s M_s^*)k_y)(x) \\ &= \sum_{\nu \in \mathbb{N}_0^3} \frac{|\nu|!}{\nu!} \left(\zeta^\nu(x)\overline{\zeta^\nu(y)} - (s \odot \zeta^\nu)(x)\overline{(s \odot \zeta^\nu)(y)} \right) \end{aligned}$$

is positive definite.

Blaschke factor (after Rudin):

Let $a \in \Omega = \left\{ x \in \mathbb{H} \mid 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1 \right\}$.

$$B_a = (1 - \zeta(a)\zeta(a)^*)^{\frac{1}{2}} (1 - \zeta\zeta(a)^*)^{-\odot} \odot (\zeta - \zeta(a)) \left(I - \zeta(a)^*\zeta(a)\right)^{-\frac{1}{2}}$$

k_1, k_2 and i the basis of the split-quaternions, with multiplication table

$\nearrow$	1	k_1	k_2	i
1	1	k_1	k_2	i
k_1	k_1	1	i	k_2
k_2	k_2	$-$i	1	$-k_1$
i	i	$-k_2$	k_1	-1

Define

$$\nabla_{\mathbb{R}}^{+} = \frac{\partial}{\partial x_0} - k_1 \frac{\partial}{\partial x_1} - k_2 \frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3},$$

and

$$\nabla_{\mathbb{R}} = \frac{\partial}{\partial x_0} + k_1 \frac{\partial}{\partial x_1} + k_2 \frac{\partial}{\partial x_2} - i \frac{\partial}{\partial x_3}$$

$$\nabla_{\mathbb{R}}^{+}\nabla_{\mathbb{R}} = \nabla_{\mathbb{R}}\nabla_{\mathbb{R}}^{+} = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Solutions of the ultra-hyperbolic wave equation $\nabla_{\mathbb{R}}^{+}\nabla_{\mathbb{R}} f = 0$ need not be smooth, let-alone real-analytic. A $H_{\mathbb{R}}$-valued real-analytic function f will be called left-regular if $\nabla_{\mathbb{R}}^{+} f = 0$.

Gleason type decomposition.

Let

$$(R_u f)(x) = \int_0^1 \frac{\partial}{\partial x_u} f(tx)dt, \quad u = 0, 1, 2, 3.$$

Let f be a left-regular function in a neighborhood of the origin. Then

$$f(x) - f(0) = \sum_{u=1}^{3} \zeta_u(x)(R_u f)(x).$$

with

$$\begin{aligned} \zeta_1(x) &= x_1 + x_0 \mathrm{k}_1 \\ \zeta_2(x) &= x_2 + x_0 \mathrm{k}_2 \\ \zeta_3(x) &= x_3 - x_0 \mathrm{i}. \end{aligned}$$

(left-regular; these functions are in fact the building blocks of all functions left-regular in a neighborhood of the origin).

Slice hyperholomorphic functions.

Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f:\ \Omega \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S} = \left\{ q \in \mathbb{H};\, q^2 = -1 \right\}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element on $\mathbb{C}_I$. We say that f is a left slice regular function if, for every $I \in \mathbb{S}$, we have:

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

Example:

The quaternionic variables and its powers are slice hyperholomorphic.

Product:

Pointwise product does not keep slice hyperholomorphy, but there is another product, which reduces to convolution on coefficients for power series, which keeps slice hyperholomorphy.

Let V be a two sided quaternionic linear space, $\mathcal{B}(V)$ denote the space of bounded right linear operators.

Definition

Let $A \in \mathcal{B}(V)$. We define the S-spectrum $\sigma_S(A)$ as
$\sigma_S(A) = \{r \in \mathbb{H} \ : \ A^2 - 2\mathrm{Re}(r)A + |r|^2 I$ is not invertible in $\mathcal{B}(V)\}$,
and the S-resolvent set $\rho_S(A) = \mathbb{H} \setminus \sigma_S(A)$.

Definition

Let $A \in \mathcal{B}(V)$ and $r \in \rho_S(A)$. We define the right S-resolvent operator as
$S_R^{-1}(r, A) := -(A - I\bar{r})(A^2 - 2\mathrm{Re}(r)A + |r|^2 I)^{-1}$.

Proposition

Let $A \in \mathcal{B}(V)$. Then, for $|p| \, \|A\| < 1$ we have

$$\sum_{n=0}^{\infty} p^n A^n = p^{-1} S_R^{-1}(p^{-1}, A) = (I - \bar{p}A)(|p|^2 A^2 - 2\mathrm{Re}(p)A + I)^{-1}.$$

Let $H_2(\mathbb{B})$ be the Hardy space of the unit ball $\mathbb{B} \subset \mathbb{H}$:

$$H_2(\mathbb{B}) = \{f(p) = \sum_{n\geq 0} p^n a_n \mid \sum_{n\geq 0} |a_n|^2 < \infty\}$$

de Branges Rovnyak spaces.

Let $S : \mathbb{B} \rightarrow \mathbb{H}$. The following are equivalent:
(1) S is a Schur function.
(2) The operator M_S of slice regular left multiplication by S

$$M_S : f \mapsto S \star f$$

is a contraction on $H_2(\mathbb{B})$.
(3) The kernel

$$K_S(p,q) = \sum_{k=0}^{\infty} p^k (1 - S(p)\overline{S(q)})\bar{q}^k$$

is positive on $\mathbb{B} \times \mathbb{B}$.

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