Eigenvalue of Linear Transformation of Vector Space over Non-Commutative Algebra

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Abstract. Let A be associative division D-algebra. Let $\overline{\overline{e}}$ be a basis of A-vector space V of columns. Let n be dimension of A-vector space V. Linear transformation of A-vector space V has form

$$\begin{pmatrix} w^1 \\ ... \\ w^n \end{pmatrix} = \begin{pmatrix} a^1_1 & ... & a^1_n \\ ... & ... & ... \\ a^n_1 & ... & a^n_n \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \qquad a^i_j \in A \otimes A$$

with respect to basis $\overline{\overline{e}}$.

A-number b is called left ${}_\circ{}^\circ$-eigenvalue of the matrix

$$a = \begin{pmatrix} a^1_1 & ... & a^1_n \\ ... & ... & ... \\ a^n_1 & ... & a^n_n \end{pmatrix}$$

if there exists column vector v such that

$$\begin{pmatrix} a^1_1 & ... & a^1_n \\ ... & ... & ... \\ a^n_1 & ... & a^n_n \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = b \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix}$$

The column vector v is called eigencolumn for left ${}_\circ{}^\circ$-eigenvalue b.

A-number b is called right ${}_\circ{}^\circ$-eigenvalue of the matrix

$$a = \begin{pmatrix} a^1_1 & ... & a^1_n \\ ... & ... & ... \\ a^n_1 & ... & a^n_n \end{pmatrix}$$

if there exists column vector v such that

$$\begin{pmatrix} a^1_1 & ... & a^1_n \\ ... & ... & ... \\ a^n_1 & ... & a^n_n \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} b$$

The column vector v is called eigencolumn for right ${}_\circ{}^\circ$-eigenvalue b.

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Solution of system of differential equations

$$\frac{dx^1}{dt}=a^1_{1\,s0}x^1a^1_{1\,s1}+...+a^1_{n\,s0}x^na^1_{n\,s1}$$
$$......$$
$$\frac{dx^n}{dt}=a^n_{1\,s0}x^1a^n_{1\,s1}+...+a^n_{n\,s0}x^na^n_{n\,s1}$$

is sum of following solutions

- $$\begin{pmatrix}x^1\\...\\x^n\end{pmatrix}=Ce^{bt}\begin{pmatrix}c^1\\...\\c^n\end{pmatrix}$$
  where A-number b is left ${}_\circ{}^\circ$-eigenvalue of the matrix
  $$a=\begin{pmatrix}a^1_{1\,s0}\otimes a^1_{1\,s1} & ... & a^1_{n\,s0}\otimes a^1_{n\,s1}\\ ... & ... & ...\\ a^n_{1\,s0}\otimes a^n_{1\,s1} & ... & a^n_{n\,s0}\otimes a^n_{n\,s1}\end{pmatrix}$$
  and the column c is eigencolumn of the matrix a corresponding to the left ${}_\circ{}^\circ$-eigenvalue b.
- $$\begin{pmatrix}x^1\\...\\x^n\end{pmatrix}=\begin{pmatrix}c^1\\...\\c^n\end{pmatrix}e^{bt}C$$
  where A-number b is right ${}_\circ{}^\circ$-eigenvalue of the matrix
  $$a=\begin{pmatrix}a^1_{1\,s0}\otimes a^1_{1\,s1} & ... & a^1_{n\,s0}\otimes a^1_{n\,s1}\\ ... & ... & ...\\ a^n_{1\,s0}\otimes a^n_{1\,s1} & ... & a^n_{n\,s0}\otimes a^n_{n\,s1}\end{pmatrix}$$
  and the column c is eigencolumn of the matrix a corresponding to the right ${}_\circ{}^\circ$-eigenvalue b.

Contents

I want to consider the method of solving the system of differential equations

$$\begin{aligned}
&\frac{dx^1}{dt} = a^{1}_{1\,s0}x^{1}a^{1}_{1\,s1} + ... + a^{1}_{n\,s0}x^{n}a^{1}_{n\,s1}\\
&......\\
&\frac{dx^n}{dt} = a^{n}_{1\,s0}x^{1}a^{n}_{1\,s1} + ... + a^{n}_{n\,s0}x^{n}a^{n}_{n\,s1}
\end{aligned} \tag{5.1}$$

In the equality (5.1), the convention on summation over the index s is adopted. Before we begin, we consider necessary definitions and theorems.

1. Helpful Theorem

Theorem 1.1. *Let A be non-commutative D-algebra. For any $b \in A$, there exists subalgebra $Z(A, b)$ of D-algebra A such that*

$$c \in Z(A, b) \Leftrightarrow cb = bc \tag{1.1}$$

D-algebra $Z(A, b)$ is called **center of A-number** *b.*

Proof. The theorem follows from the theorem [2]-5.1.10. □

[2] Aleks Kleyn, Introduction into Noncommutative Algebra, Volume 1, Division Algebra
eprint arXiv:2207.06506 (2022)

Theorem 1.2. *Let A be non-commutative D-algebra. For any $a \in A$, if $c \in Z(A, a)$, then*

$$p(c) \in Z(A, a) \tag{1.2}$$

for any polynomial

$$\begin{aligned} p(x) &= p_0 + p_1x + ... + p_nx^n\\ p_0&,\ ...,\ p_n \in D \end{aligned} \tag{1.3}$$

Theorem 1.3. *Let A be Banach associative D-algebra and $a, c \in A$. The condition*

$$c \in Z(A, a) \tag{1.4}$$

implies that

$$e^{at}c = ce^{at} \tag{1.5}$$

Proof. The theorem follows from the theorem [1]-20.1.7. □

[1] Aleks Kleyn, Differential Equation over Banach Algebra,
eprint arXiv:1801.01628 (2018)

2. Matrix of A-Numbers

I recall that there are two operations of product of matrices with entries from non-commutative algebra A.

Definition 2.1. *Let the nubmer of columns of the matrix a equal the number of rows of the matrix b.* **${}_*{}^*$-product** *of matrices a and b has form*

$$a{}_*{}^*b = \left(a_k^i b_j^k\right) \tag{2.1}$$

$$(a{}_*{}^*b)_j^i = a_k^i b_j^k \tag{2.2}$$

$$\begin{pmatrix} a_1^1 & ... & a_p^1 \\ ... & ... & ... \\ a_1^n & ... & a_p^n \end{pmatrix} {}_*{}^* \begin{pmatrix} b_1^1 & ... & b_m^1 \\ ... & ... & ... \\ b_1^p & ... & b_m^p \end{pmatrix} = \begin{pmatrix} a_k^1 b_1^k & ... & a_k^1 b_m^k \\ ... & ... & ... \\ a_k^n b_1^k & ... & a_k^n b_m^k \end{pmatrix} = \begin{pmatrix} (a{}_*{}^*b)_1^1 & ... & (a{}_*{}^*b)_m^1 \\ ... & ... & ... \\ (a{}_*{}^*b)_1^n & ... & (a{}_*{}^*b)_m^n \end{pmatrix} \tag{2.3}$$

${}_*{}^*$-product can be expressed as product of a row of the matrix a over a column of the matrix b. □

Definition 2.2. *Let the nubmer of rows of the matrix a equal the number of columns of the matrix b.* **${}^*{}_*$-product** *of matrices a and b has form*

$$a{}^*{}_*b = \left(a_i^k b_k^j\right) \tag{2.4}$$

$$(a{}^*{}_*b)_j^i = a_i^k b_k^j \tag{2.5}$$

$$\begin{pmatrix} a_1^1 & ... & a_m^1 \\ ... & ... & ... \\ a_1^p & ... & a_m^p \end{pmatrix} {}^*{}_* \begin{pmatrix} b_1^1 & ... & b_p^1 \\ ... & ... & ... \\ b_1^n & ... & b_p^n \end{pmatrix} = \begin{pmatrix} a_1^k b_k^1 & ... & a_m^k b_k^1 \\ ... & ... & ... \\ a_1^k b_k^n & ... & a_m^k b_k^n \end{pmatrix} = \begin{pmatrix} (a{}^*{}_*b)_1^1 & ... & (a{}^*{}_*b)_m^1 \\ ... & ... & ... \\ (a{}^*{}_*b)_1^n & ... & (a{}^*{}_*b)_m^n \end{pmatrix} \tag{2.6}$$

${}^*{}_*$-product can be expressed as product of a column of the matrix a over a row of the matrix b. □

In following definitions, we consider different types of eigenvalues of matrix of A-numbers.

Definition 2.3. *A-number b is called* **${}_*{}^*$-eigenvalue** *of the matrix f if the matrix $f - bE_n$ is ${}_*{}^*$-singular matrix.* □

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Definition 2.5. *Let A-number b be ${}_*{}^*$-eigenvalue of the matrix f. A column v is called* **eigencolumn** *of matrix f corresponding to ${}_*{}^*$-eigenvalue b, if the following equality is true*

$$f{}_*{}^*v = bv \tag{2.7}$$

□

Definition 2.6. *Let A-number b be ${}^*{}_*$-eigenvalue of the matrix f. A column v is called* **eigencolumn** *of matrix f corresponding to ${}^*{}_*$-eigenvalue b, if the following equality is true*

$$v{}^*{}_*f = vb \tag{2.8}$$

□

Definition 2.7. *Let a_2 be $n \times n$ matrix which is ${}^*{}_*$-similar to diagonal matrix a_1*

$$a_1 = \mathrm{diag}(b(1), ..., b(n))$$

Thus, there exist non-${}^*{}_*$-singular matrix u_2 such that

$$u_2{}^*{}_*a_2{}^*{}_*u_2^{-1^*{}_*} = a_1 \tag{2.9}$$

$$u_2{}^*{}_*a_2 = a_1{}^*{}_*u_2 \tag{2.10}$$

The column u_{2i} of the matrix u_2 satisfies to the equality

$$u_{2i}{}^*{}_*a_2 = b(i)u_{2i} \tag{2.11}$$

*The A-number $b(i)$ is called left ${}^*{}_*$-***eigenvalue** *and column vector u_{2i} is called* **eigencolumn** *for left ${}^*{}_*$-eigenvalue $b(i)$.* □

Definition 2.8. *Let a_2 be $n \times n$ matrix which is ${}_*{}^*$-similar to diagonal matrix a_1*

$$a_1 = \mathrm{diag}(b(1), ..., b(n))$$

Thus, there exist non-${}_*{}^*$-singular matrix u_2 such that

$$u_2^{-1_*{}^*}{}_*{}^*a_2{}_*{}^*u_2 = a_1 \tag{2.12}$$

$$a_2{}_*{}^*u_2 = u_2{}_*{}^*a_1 \tag{2.13}$$

The column u_{2i} of the matrix u_2 satisfies to the equality

$$a_2{}_*{}^*u_{2i} = u_{2i}b(i) \tag{2.14}$$

*The A-number $b(i)$ is called right ${}_*{}^*$-***eigenvalue** *and column vector u_{2i} is called* **eigencolumn** *for right ${}_*{}^*$-eigenvalue $b(i)$.* □

3. Linear map of A-vector space

Let A be associative division D-algebra. We consider D-algebra A which has center $Z(A) = D$.

Theorem 3.1. *We can identify linear map*

$$a : A \to A$$

of D-algebra A and tensor

$$a_{s0} \otimes a_{s1} \in A^{2\otimes} \tag{3.1}$$

by the equality

$$a \circ x = (a_{s0} \otimes a_{s1}) \circ x = a_{s0}xa_{s1} \tag{3.2}$$

Let linear map

$$a : A \to A$$

have representation

$$a \circ x = (a_{s0} \otimes a_{s1}) \circ x = a_{s0}xa_{s1} \tag{3.3}$$

and linear map

$$b : A \to A$$

have representation

$$b \circ x = (b_{t0} \otimes b_{t1}) \circ x = b_{t0} x b_{t1} \tag{3.4}$$

Then product of maps a and b has the following form

$$\begin{aligned} a \circ b \circ x &= ((a_{s0} \otimes a_{s1}) \circ (b_{t0} \otimes b_{t1})) \circ x \\ &= ((a_{s0} b_{t0}) \otimes (b_{t1} a_{s1})) \circ x \\ &= a_{s0} b_{t0} x b_{t1} a_{s1} \end{aligned} \tag{3.5}$$

Definition 3.2. *Let a be a matrix and $a^i_j \in A^{n\otimes}$. The matrix a is called matrix of tensors $A^{n\otimes}$.* □

The product of maps

$$a \circ b = (a_{s0} \otimes a_{s1}) \circ (b_{t0} \otimes b_{t1}) = (a_{s0} b_{t0}) \otimes (b_{t1} a_{s1}) \tag{3.6}$$

discussed above can be extended to product of matrices of maps $(a = (a^i_j),\ a^i_j \in A^{2\otimes})$.

Definition 3.3. *Let $a^i_j \in A^{2\otimes}$, $b^i_j \in A^{2\otimes}$. We introduce ${}_\circ{}^\circ$-product of matrices of maps*

$$\begin{pmatrix} a^1_1 & ... & a^1_n \\ ... & ... & ... \\ a^m_1 & ... & a^m_n \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} b^1_1 & ... & b^1_k \\ ... & ... & ... \\ b^n_1 & ... & b^n_k \end{pmatrix} = \begin{pmatrix} a^1_i \circ b^i_1 & ... & a^1_i \circ b^i_k \\ ... & ... & ... \\ a^m_i \circ b^i_1 & ... & a^m_i \circ b^i_k \end{pmatrix} \tag{3.7}$$

using the following equality

$$(a{}_\circ{}^\circ b)^i_j = a^i_k \circ b^k_j \tag{3.8}$$

□

Remark 3.4. *Linear map of vector space V over field D is homomorphism of D-vector space V. Therefore, we use a matrix of D-numbers as coordinate representation of linear map or homomorphism.*

If we consider vector space V over division D-algebra A, then considered similarity between linear map and homomorphism will be broken. We still use a matrix of A-numbers to represent homomorphis of A-vector space. However we cannot confine ourselves to the set of homomorphisms to consider linear maps because in this case we will not be able to define polylinear map. This is why we define linear map of A-vector space V as linear map of corresponding D-vector space V. Therefore, for a linear map it does not matter we consider left or right A-vector space. □

Theorem 3.5. *Let $V^1, ..., V^n$, $W^1, ..., W^m$ be A-vector spaces and*

$$V = V^1 \oplus ... \oplus V^n$$

$$W = W^1 \oplus ... \oplus W^m$$

Let us represent V-number

$$v = v^1 \oplus ... \oplus v^n$$

as column vector

$$v = \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \tag{3.9}$$

Let us represent W-number

$$w = w^1 \oplus ... \oplus w^m$$

as column vector

$$w = \begin{pmatrix} w^1 \\ ... \\ w^m \end{pmatrix} \tag{3.10}$$

Then the linear map

$$f : V \to W$$

has representation as a matrix of maps

$$f = \begin{pmatrix} f^1_1 & ... & f^1_n \\ ... & ... & ... \\ f^m_1 & ... & f^m_n \end{pmatrix} \tag{3.11}$$

such way that, if $w = f \circ v$, *then*

$$\begin{pmatrix} w^1 \\ ... \\ w^m \end{pmatrix} = \begin{pmatrix} f^1_1 & ... & f^1_n \\ ... & ... & ... \\ f^m_1 & ... & f^m_n \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = \begin{pmatrix} f^1_i \circ v^i \\ ... \\ f^m_i \circ v^i \end{pmatrix} \tag{3.12}$$

The map

$$f^i_j : V^j \to W^i$$

is a linear map and is called **partial linear map**.

Proof. The theorem follows from the theorem [2]-7.5.12. □

Let V be left A-vector space of columns of dimension n. Let $\overline{\overline{e}} = (e_1, ..., e_n)$ be a basis of A-vector space V. For any i, the set $A_i = Ae_i$ is subspace of A-vector space V. A-vector space V is direct sum of A-vector spaces $A_i = Ae_i$

$$V = Ae_1 \oplus ... \oplus Ae_n \tag{3.13}$$

Let V be right A-vector space of columns of dimension n. Let $\overline{\overline{e}} = (e_1, ..., e_n)$ be a basis of A-vector space V. For any i, the set $A_i = e_iA$ is subspace of A-vector space V. A-vector space V is direct sum of A-vector spaces $A_i = e_iA$

$$V = e_1A \oplus ... \oplus e_nA \tag{3.14}$$

Linear map

$$f : V \to V$$

of A-vector space V is called linear transformation of A-vector space V. According to the theorem 3.5, linear transformation of A-vector space V has the following form

$$a\circ\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}=\begin{pmatrix}a_1^1&...&a_n^1\\ ...&...&...\\ a_1^n&...&a_n^n\end{pmatrix}{}_\circ{}^\circ\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}\tag{3.15}$$

where the map

$$a_j^i:A_j\to A_i$$

is partial linear map.

4. Eigenvalue of Matrix of Linear Map

Definition 4.1. *A-number b is called left ${}_\circ{}^\circ$-***eigenvalue** *of the matrix a if there exists column vector v such that*

$$a_\circ{}^\circ v=bv\tag{4.1}$$

The column vector v is called **eigencolumn** *for left ${}_\circ{}^\circ$-eigenvalue b.* □

Definition 4.2. *A-number b is called right ${}_\circ{}^\circ$-***eigenvalue** *of the matrix a if there exists column vector v such that*

$$a_\circ{}^\circ v=vb\tag{4.2}$$

The column vector v is called **eigencolumn** *for right ${}_\circ{}^\circ$-eigenvalue b.* □

Theorem 4.3. *Let entries of the matrix a satisfy the equality*

$$a_j^i{}_{s0}\otimes a_j^i{}_{s1}=1\otimes a_1{}_j^i\tag{4.3}$$

Then left ${}_\circ{}^\circ$-eigenvalue b is left ${}^*{}_*$-eigenvalue of the matrix a_1.

Theorem 4.4. *Let entries of the matrix a satisfy the equality*

$$a_j^i{}_{s0}\otimes a_j^i{}_{s1}=a_0{}_j^i\otimes 1\tag{4.4}$$

Then right ${}_\circ{}^\circ$-eigenvalue b is right ${}_*{}^*$-eigenvalue of the matrix a_0.

Proof of theorem 4.3. The equality

$$(a_j^i{}_{s0}\otimes a_j^i{}_{s1})\circ v^j=(1\otimes a_1{}_j^i)\circ v^j=v^ja_1{}_j^i\tag{4.5}$$

follows from the equality $\boxed{(4.3)\qquad a_j^i{}_{s0}\otimes a_j^i{}_{s1}=1\otimes a_1{}_j^i}$.

The equality

$$\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}{}^*{}_*\begin{pmatrix}a_1{}_1^1&...&a_1{}_n^1\\ ...&...&...\\ a_1{}_1^n&...&a_1{}_n^n\end{pmatrix}=\begin{pmatrix}a_1^1{}_{s0}\otimes a_1^1{}_{s1}&...&a_n^1{}_{s0}\otimes a_n^1{}_{s1}\\ ...&...&...\\ a_1^n{}_{s0}\otimes a_1^n{}_{s1}&...&a_n^n{}_{s0}\otimes a_n^n{}_{s1}\end{pmatrix}{}_\circ{}^\circ\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}\tag{4.6}$$

follows from the equality (4.5). The equality

$$\begin{pmatrix}a_1^1{}_{s0}\otimes a_1^1{}_{s1}&...&a_n^1{}_{s0}\otimes a_n^1{}_{s1}\\ ...&...&...\\ a_1^n{}_{s0}\otimes a_1^n{}_{s1}&...&a_n^n{}_{s0}\otimes a_n^n{}_{s1}\end{pmatrix}{}_\circ{}^\circ\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}=b\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}\tag{4.7}$$

follows from the definition 4.1 of left ${}_\circ{}^\circ$-eigenvalue of the matrix a. The equality

$$\begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} {}^*{}_* \begin{pmatrix} {a_1}^1_1 & ... & {a_1}^1_n \\ ... & ... & ... \\ {a_1}^n_1 & ... & {a_1}^n_n \end{pmatrix} = b \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \tag{4.8}$$

follows from equalities (4.6), (4.7). The theorem follows from the equality (4.8) and from the definition 2.7 of left ${}^*{}_*$-eigenvalue of the matrix a_1. □

Proof of theorem 4.4. The equality

$$(a^i_{j\,s0} \otimes a^i_{j\,s1}) \circ v^j = ({a_0}^i_j \otimes 1) \circ v^j = {a_0}^i_j v^j \tag{4.9}$$

follows from the equality $\boxed{(4.4) \qquad a^i_{j\,s0} \otimes a^i_{j\,s1} = {a_0}^i_j \otimes 1}$.

The equality

$$\begin{pmatrix} {a_0}^1_1 & ... & {a_0}^1_n \\ ... & ... & ... \\ {a_0}^n_1 & ... & {a_0}^n_n \end{pmatrix} {}_*{}^* \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = \begin{pmatrix} a^1_{1\,s0} \otimes a^1_{1\,s1} & ... & a^1_{n\,s0} \otimes a^1_{n\,s1} \\ ... & ... & ... \\ a^n_{1\,s0} \otimes a^n_{1\,s1} & ... & a^n_{n\,s0} \otimes a^n_{n\,s1} \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \tag{4.10}$$

follows from the equality (4.9). The equality

$$\begin{pmatrix} a^1_{1\,s0} \otimes a^1_{1\,s1} & ... & a^1_{n\,s0} \otimes a^1_{n\,s1} \\ ... & ... & ... \\ a^n_{1\,s0} \otimes a^n_{1\,s1} & ... & a^n_{n\,s0} \otimes a^n_{n\,s1} \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} b \tag{4.11}$$

follows from the definition 4.2 of right ${}_\circ{}^\circ$-eigenvalue of the matrix a. The equality

$$\begin{pmatrix} {a_0}^1_1 & ... & {a_0}^1_n \\ ... & ... & ... \\ {a_0}^n_1 & ... & {a_0}^n_n \end{pmatrix} {}_*{}^* \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} b \tag{4.12}$$

follows from equalities (4.10), (4.11). The theorem follows from the equality (4.12) and from the definition 2.8 of right ${}_*{}^*$-eigenvalue of the matrix a_0. □

Theorem 4.5. *Let entries of the matrix a satisfy the equality*

$$a^i_{j\,s0} \otimes a^i_{j\,s1} = {a_0}^i_j \otimes 1 \tag{4.13}$$

Then left ${}_\circ{}^\circ$-eigenvalue b is ${}_*{}^*$-eigenvalue of the matrix a_0.

Theorem 4.6. *Let entries of the matrix a satisfy the equality*

$$a^i_{j\,s0} \otimes a^i_{j\,s1} = 1 \otimes {a_1}^i_j \tag{4.14}$$

Then right ${}_\circ{}^\circ$-eigenvalue b is ${}^*{}_*$-eigenvalue of the matrix a_1.

Proof of theorem 4.5. The equality

$$(a^i_{j\,s0} \otimes a^i_{j\,s1}) \circ v^j = ({a_0}^i_j \otimes 1) \circ v^j = {a_0}^i_j v^j \tag{4.15}$$

follows from the equality $\boxed{(4.13) \qquad a^i_{j\,s0} \otimes a^i_{j\,s1} = {a_0}^i_j \otimes 1}$.

The equality

$$\begin{pmatrix} a_0{}^1_1 & ... & a_0{}^1_n \\ ... & ... & ... \\ a_0{}^n_1 & ... & a_0{}^n_n \end{pmatrix} {}_*{}^* \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = \begin{pmatrix} a^1_1{}_{s0} \otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0} \otimes a^1_n{}_{s1} \\ ... & ... & ... \\ a^n_1{}_{s0} \otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0} \otimes a^n_n{}_{s1} \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \tag{4.16}$$

follows from the equality (4.15). The equality

$$\begin{pmatrix} a^1_1{}_{s0} \otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0} \otimes a^1_n{}_{s1} \\ ... & ... & ... \\ a^n_1{}_{s0} \otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0} \otimes a^n_n{}_{s1} \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = b \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \tag{4.17}$$

follows from the definition 4.1 of left ${}_\circ{}^\circ$-eigenvalue of the matrix a. The equality

$$\begin{pmatrix} a_0{}^1_1 & ... & a_0{}^1_n \\ ... & ... & ... \\ a_0{}^n_1 & ... & a_0{}^n_n \end{pmatrix} {}_*{}^* \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = b \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \tag{4.18}$$

follows from equalities (4.16), (4.17). The theorem follows from the equality (4.18), from the definition 2.3 of ${}^*{}_*$-eigenvalue of the matrix a_0 and from the definition 2.5 of corresponding eigenvector v. □

Proof of theorem 4.6. The equality

$$(a^i_j{}_{s0} \otimes a^i_j{}_{s1}) \circ v^j = (1 \otimes a_1{}^i_j) \circ v^j = v^j a_1{}^i_j \tag{4.19}$$

follows from the equality $\boxed{(4.14) \quad a^i_j{}_{s0} \otimes a^i_j{}_{s1} = 1 \otimes a_1{}^i_j}$.

The equality

$$\begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} {}^*{}_* \begin{pmatrix} a_1{}^1_1 & ... & a_1{}^1_n \\ ... & ... & ... \\ a_1{}^n_1 & ... & a_1{}^n_n \end{pmatrix} = \begin{pmatrix} a^1_1{}_{s0} \otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0} \otimes a^1_n{}_{s1} \\ ... & ... & ... \\ a^n_1{}_{s0} \otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0} \otimes a^n_n{}_{s1} \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} \tag{4.20}$$

follows from the equality (4.19). The equality

$$\begin{pmatrix} a^1_1{}_{s0} \otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0} \otimes a^1_n{}_{s1} \\ ... & ... & ... \\ a^n_1{}_{s0} \otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0} \otimes a^n_n{}_{s1} \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} = \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} b \tag{4.21}$$

follows from the definition 4.2 of right ${}_\circ{}^\circ$-eigenvalue of the matrix a. The equality

$$\begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} {}^*{}_* \begin{pmatrix} a_1{}^1_1 & ... & a_1{}^1_n \\ ... & ... & ... \\ a_1{}^n_1 & ... & a_1{}^n_n \end{pmatrix} = \begin{pmatrix} v^1 \\ ... \\ v^n \end{pmatrix} b \tag{4.22}$$

follows from equalities (4.20), (4.21). The theorem follows from the equality (4.22), from the definition 2.4 of ${}_*{}^*$-eigenvalue of the matrix a_1 and from the definition 2.6 of corresponding eigenvector v. □

Theorem 4.7. *Let the column vector v be eigencolumn for left ${}_\circ{}^\circ$-eigenvalue b of the matrix a. Let A-number b satisfy the condition*

$$b \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} Z(A, a^i_j{}_{s0}) \tag{4.23}$$

Then for any polynomial

$$\begin{aligned} p(x) &= p_0 + p_1 x + ... + p_n x^n \\ p_0,\ &...,\ p_n \in D \end{aligned} \tag{4.24}$$

the column vector

$$cv = \begin{pmatrix} cv^1 \\ ... \\ cv^n \end{pmatrix}$$

$$c = p(b)$$

is also eigencolumn for left ${}_\circ{}^\circ$-eigenvalue b.

Proof. The theorem follows from the theorem [2]-14.4.7. □

Theorem 4.8. *Let the column vector v be eigencolumn for right ${}_\circ{}^\circ$-eigenvalue b of the matrix a. Let A-number b satisfy the condition*

$$b \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} Z(A, a^i_j{}_{s1}) \tag{4.25}$$

Then for any polynomial

$$\begin{aligned} p(x) &= p_0 + p_1 x + ... + p_n x^n \\ p_0,\ &...,\ p_n \in D \end{aligned} \tag{4.26}$$

the column vector

$$vc = \begin{pmatrix} v^1 c \\ ... \\ v^n c \end{pmatrix}$$

$$c = p(b)$$

is also eigencolumn for right ${}_\circ{}^\circ$-eigenvalue b.

Proof. The theorem follows from the theorem [2]-14.4.8. □

5. Differential Equation $\dfrac{dx}{dt} = a{}_\circ{}^\circ x$

We can represent the system of differential equations

$$\begin{aligned} \frac{dx^1}{dt} &= a^1_1{}_{s0} x^1 a^1_1{}_{s1} + ... + a^1_n{}_{s0} x^n a^1_n{}_{s1} \\ &...... \\ \frac{dx^n}{dt} &= a^n_1{}_{s0} x^1 a^n_1{}_{s1} + ... + a^n_n{}_{s0} x^n a^n_n{}_{s1} \end{aligned} \tag{5.1}$$

using product of matrices

$$\begin{pmatrix} \dfrac{dx^1}{dt} \\ ... \\ \dfrac{dx^n}{dt} \end{pmatrix} = \begin{pmatrix} a^1_1{}_{s0} \otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0} \otimes a^1_n{}_{s1} \\ ... & ... & ... \\ a^n_1{}_{s0} \otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0} \otimes a^n_n{}_{s1} \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} x^1 \\ ... \\ x^n \end{pmatrix} \tag{5.2}$$

Theorem 5.1. *Let A-number b be left ${}_\circ{}^\circ$-eigenvalue of the matrix*

$$a = \begin{pmatrix} a^1_1{}_{s0} \otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0} \otimes a^1_n{}_{s1} \\ ... & ... & ... \\ a^n_1{}_{s0} \otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0} \otimes a^n_n{}_{s1} \end{pmatrix}$$

Theorem 5.2. *Let A-number b be right ${}_\circ{}^\circ$-eigenvalue of the matrix*

$$a = \begin{pmatrix} a^1_1{}_{s0} \otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0} \otimes a^1_n{}_{s1} \\ ... & ... & ... \\ a^n_1{}_{s0} \otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0} \otimes a^n_n{}_{s1} \end{pmatrix}$$

and satisfies the condition

$$b \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} Z(A, a^{i}_{j\,s0}) \tag{5.3}$$

Then the system of differential equations (5.2) *has the solution*

$$x = Ce^{bt}c = Ce^{bt}\begin{pmatrix} c^1 \\ ... \\ c^n \end{pmatrix} \tag{5.4}$$

where A-number C satisfies the condition

$$C \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} Z(A, a^{i}_{j\,s0}) \tag{5.5}$$

and the column c is eigencolumn of the matrix a corresponding to the left ${}_\circ{}^\circ$*-eigenvalue b.*

Proof. The theorem follows from the theorem [1]-21.5.1. □

and satisfies the condition

$$b \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} Z(A, a^{i}_{j\,s1}) \tag{5.6}$$

Then the system of differential equations (5.2) *has the solution*

$$x = ce^{bt}C = \begin{pmatrix} c^1 \\ ... \\ c^n \end{pmatrix} e^{bt}C \tag{5.7}$$

where A-number C satisfies the condition

$$C \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} Z(A, a^{i}_{j\,s1}) \tag{5.8}$$

and the column c is eigencolumn of the matrix a corresponding to the right ${}_\circ{}^\circ$*-eigenvalue b.*

Proof. The theorem follows from the theorem [1]-21.5.2. □

Consider differential equation

$$\begin{aligned}(1\otimes 1)\circ \frac{d^n y}{dt^n} + (a_{n\,s0}\otimes a_{n\,s1})\circ \frac{d^{n-1}y}{dt^{n-1}} + ...\\ + (a_{2\,s0}\otimes a_{2\,s1})\circ \frac{dy}{dt} + (a_{1\,s0}\otimes a_{1\,s1})\circ y = 0\end{aligned} \tag{5.9}$$

Theorem 5.3. *Let a map* $x = e^{bt}$ *be solution of differential equation* (5.9)*. Then A-number b is a root of the equation*

$$x^n + a_{n\,s0}x^{n-1}a_{n\,s1} + ... + a_{2\,s0}xa_{2\,s1} + a_{1\,s0}a_{1\,s1} = 0 \tag{5.10}$$

and satisfies either the condition

$$b \in \bigcap_{i=1}^{n} Z(A, a_{i\,s0}) \tag{5.11}$$

or the condition

$$b \in \bigcap_{i=1}^{n} Z(A, a_{i\,s1}) \tag{5.12}$$

The differential equation

$$\frac{d^2x}{dt^2} - i\frac{dx}{dt} - \frac{dx}{dt}j + ixj = 0 \tag{5.13}$$

is equivalent to the system of differential equations

$$\begin{aligned}\frac{dx^1}{dt} &= x^2\\ \frac{dx^2}{dt} &= ix^2 + x^2 j - ix^1 j\end{aligned} \tag{5.14}$$

The matrix representation of the system of differential equations (5.14) has the following form

$$\begin{pmatrix}\dfrac{dx^1}{dt}\\ \\ \dfrac{dx^2}{dt}\end{pmatrix} = \begin{pmatrix}0\otimes 0 & 1\otimes 1\\ -i\otimes j & i\otimes 1 + 1\otimes j\end{pmatrix} {}_\circ{}^\circ \begin{pmatrix}x^1\\ x^2\end{pmatrix} \tag{5.15}$$

We have yet to find a method for determining eigenvalues of the matrix

$$a = \begin{pmatrix} a^1_1{}_{s0}\otimes a^1_1{}_{s1} & ... & a^1_n{}_{s0}\otimes a^1_n{}_{s1}\\ ... & ... & ...\\ a^n_1{}_{s0}\otimes a^n_1{}_{s1} & ... & a^n_n{}_{s0}\otimes a^n_n{}_{s1}\end{pmatrix} \tag{5.16}$$

However according to the theorem 5.3, eigenvalues of matrix

$$\begin{pmatrix}0\otimes 0 & 1\otimes 1\\ -i\otimes j & i\otimes 1 + 1\otimes j\end{pmatrix} \tag{5.17}$$

are roots of the equation

$$x^2 - ix - xj + k = 0 \tag{5.18}$$

- The value

$$b = i \tag{5.19}$$

satisfies the condition (5.3) $b \in \bigcap_{i=1}^{n}\bigcap_{j=1}^{n} Z(A, a^i_j{}_{s0})$.

$a^1_1{}_{10} = 0$, $a^1_2{}_{10} = 1$, $a^2_1{}_{10} = -i$, $a^2_2{}_{10} = i$, $a^2_2{}_{20} = 1$.

According to the theorem 5.1, we are looking for a solution in the form

$$\begin{pmatrix}x^1\\ x^2\end{pmatrix} = (C_i^0 + C_i^1 i)e^{it}\begin{pmatrix}c_i^1\\ c_i^2\end{pmatrix} \tag{5.20}$$

where

* $C_i^0, C_i^1 \in R$
* $C_i = C_i^0 + C_i^1 i$ satisfies the condition (5.5) $C_i \in \bigcap_{i=1}^{n}\bigcap_{j=1}^{n} Z(A, a^i_j{}_{s0})$
* The column c_i is eigencolumn of the matrix a corresponding to the left ${}_\circ{}^\circ$-eigenvalue $b = i$.

According to theorems 1.2, 1.3,

$$(C_i^0 + C_i^1 i)e^{it} = e^{it}(C_i^0 + C_i^1 i) \tag{5.21}$$

The equality

$$\begin{pmatrix} x^{1} \\ x^{2} \end{pmatrix} = e^{it}(C_i^0 + C_i^1 i) \begin{pmatrix} c_i^{1} \\ c_i^{2} \end{pmatrix} \tag{5.22}$$

follows from the equality (5.20), (5.21). According to the theorem 4.7, the column $C_i c_i$ is eigencolumn of the matrix a corresponding to the left ${}_\circ{}^\circ$-eigenvalue $b = i$. Since for us it does not matter the format of notation of eigencolumn of matrix a, we can represent the solution of the system of differential equations

$$(5.14) \qquad \frac{dx^{1}}{dt} = x^{2} \qquad \frac{dx^{2}}{dt} = ix^{2} + x^{2} j - ix^{1} j$$

as

$$\begin{pmatrix} x^{1} \\ x^{2} \end{pmatrix} = e^{it} \begin{pmatrix} c_i^{1} \\ c_i^{2} \end{pmatrix} \tag{5.23}$$

The equality

$$\begin{pmatrix} 0 \otimes 0 & 1 \otimes 1 \\ -i \otimes j & i \otimes 1 + 1 \otimes j \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} c_i^{1} \\ c_i^{2} \end{pmatrix} = i \begin{pmatrix} c_i^{1} \\ c_i^{2} \end{pmatrix} \tag{5.24}$$

follows from equalities

$$(4.1) \qquad a_\circ{}^\circ v = bv \qquad\qquad (5.19) \qquad b = i$$

Equalities

$$c_i^{2} = ic_i^{1} \tag{5.25}$$

$$-ic_i^{1} j + ic_i^{2} + c_i^{2} j = ic_i^{2} \tag{5.26}$$

follow from the equality (5.24) and the equality

$$(3.8) \qquad (a_\circ{}^\circ b)_j^i = a_k^i \circ b_j^k$$

The equality $(5.26) \quad -ic_i^{1} j + ic_i^{2} + c_i^{2} j = ic_i^{2}$ follows from the equality $(5.25) \quad c_i^{2} = ic_i^{1}$. Therefore, the solution is (we set $d_i = c_i^{1}$)

$$\begin{pmatrix} x^{1} \\ x^{2} \end{pmatrix} = e^{it} \begin{pmatrix} d_i \\ id_i \end{pmatrix} \tag{5.27}$$

It remains to check the solution

$$\begin{aligned}\frac{dx^1}{dt} &= e^{it}id_i = x^2\\ \frac{dx^2}{dt} &= e^{it}iid_i = -e^{it}d_i =\\ -ix^1j + ix^2 + x^2j &= -ie^{it}d_ij + ie^{it}id_i + e^{it}id_ij\\ &= -ie^{it}d_ij + iie^{it}d_i + ie^{it}d_ij\\ &= -e^{it}d_i\end{aligned} \tag{5.28}$$

- The value

$$b = j \tag{5.29}$$

satisfies the condition $\boxed{(5.6)\qquad b \in \bigcap_{i=1}^{n}\bigcap_{j=1}^{n} Z(A, a^i_{j\,s1})}$.

$a^1_1{}_{11} = 0$, $a^1_2{}_{11} = 1$, $a^2_1{}_{11} = j$, $a^2_2{}_{11} = 1$, $a^2_2{}_{21} = j$.

According to the theorem 5.2, we are looking for a solution in the form

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} c^1_j \\ c^2_j \end{pmatrix} e^{jt}(C^0_j + C^1_j j) \tag{5.30}$$

where

* $C^0_i,\ C^1_i \in R$
* $C_j = C^0_j + C^1_j j$ satisfies the condition

$$\boxed{(5.8)\qquad C_j \in \bigcap_{i=1}^{n}\bigcap_{j=1}^{n} Z(A, a^i_{j\,s1})}$$

* The column c_j is eigencolumn of the matrix a corresponding to the right ${}_\circ{}^\circ$-eigenvalue $b = j$.

According to theorems 1.2, 1.3,

$$(C^0_j + C^1_j j)e^{jt} = e^{jt}(C^0_j + C^1_j j) \tag{5.31}$$

The equality

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} c^1_j \\ c^2_j \end{pmatrix} (C^0_j + C^1_j j)e^{jt} \tag{5.32}$$

follows from the equality (5.30), (5.31). According to the theorem 4.8, the column c_jC_j is eigencolumn of the matrix a corresponding to the right ${}_\circ{}^\circ$-eigenvalue $b = j$. Since for us it does not matter the format of notation of eigencolumn of matrix a, we can represent the solution of the system of differential equations

$$\boxed{(5.14)\qquad \begin{aligned}\frac{dx^1}{dt} &= x^2\\ \frac{dx^2}{dt} &= ix^2 + x^2j - ix^1j\end{aligned}}$$

as

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} c_j^1 \\ c_j^2 \end{pmatrix} e^{jt} \tag{5.33}$$

The equality

$$\begin{pmatrix} 0\otimes 0 & 1\otimes 1 \\ -i\otimes j & i\otimes 1+1\otimes j \end{pmatrix} {}_\circ{}^\circ \begin{pmatrix} c_j^1 \\ c_j^2 \end{pmatrix} = \begin{pmatrix} c_j^1 \\ c_j^2 \end{pmatrix} j \tag{5.34}$$

follows from equalities

(4.2) $a_\circ{}^\circ v = vb$	(5.29) $b = j$

Equalities

$$c_j^2 = c_j^1 j \tag{5.35}$$

$$-ic_j^1 j + ic_j^2 + c_j^2 j = c_j^2 j \tag{5.36}$$

follow from the equality (5.34) and the equality

(3.8) $(a_\circ{}^\circ b)_j^i = a_k^i \circ b_j^k$

The equality (5.36) $-ic_j^1 j + ic_j^2 + c_j^2 j = c_j^2 j$ follows from the equality (5.35) $c_j^2 = c_j^1 j$. Therefore, the solution is (we set $d_j = c_j^1$)

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} d_j \\ d_j j \end{pmatrix} e^{jt} \tag{5.37}$$

It remains to check the solution

$$\begin{aligned} \frac{dx^1}{dt} &= d_j j e^{jt} = x^2 \\ \frac{dx^2}{dt} &= d_j j j e^{jt} = -d_j e^{jt} = \\ -ix^1 j + ix^2 + x^2 j &= -id_j e^{jt} j + id_j j e^{jt} + d_j j e^{jt} j \\ &= -id_j e^{jt} j + id_j e^{jt} j + d_j e^{jt} j j \\ &= -d_j e^{jt} \end{aligned} \tag{5.38}$$

Therefore, general solution of the system of differential equations

$$\frac{dx^1}{dt} = x^2 \qquad \frac{dx^2}{dt} = ix^2 + x^2 j - ix^1 j \tag{5.14}$$

is

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = e^{it} \begin{pmatrix} d_i \\ id_i \end{pmatrix} + \begin{pmatrix} d_j \\ d_j j \end{pmatrix} e^{jt} \tag{5.39}$$

The general solution of the differential equation

$$\frac{d^2x}{dt^2} - i\frac{dx}{dt} - \frac{dx}{dt} j + ixj = 0 \tag{5.13}$$

is

$$x = e^{it}d_i + d_j e^{jt} \tag{5.40}$$

Remark 5.4. *Product and division of polynomials in non-commutative algebra are not simple operations. So Ore suggested to consider left-sided (right-sided) polynomials. Ore defined product of left-sided polynomials in such a way that the result of the operation is left-sided polynomial. Ore also proposed a method of transformation of polynomial into left-sided polynomial. This algebra is called Ore polynomial ring.*

Method for solving differential equation (5.13) raises the question of the limits of applicability of Ore polynomial ring because we cannot find both roots of the equation (5.18) $x^2 - ix - xj + k = 0$ *by transforming the polynomial (5.18) into left-sided or right-sided polynomial.* □

See the definition of Ore polynomial ring in the chapter [3]-2.

[3] Paul M. Cohn, Skew Fields, Cambridge University Press, 1995

6. References

[1] Aleks Kleyn, Differential Equation over Banach Algebra, eprint arXiv:1801.01628 (2018)

[2] Aleks Kleyn, Introduction into Noncommutative Algebra, Volume 1, Division Algebra eprint arXiv:2207.06506 (2022)

[3] Paul M. Cohn, Skew Fields, Cambridge University Press, 1995

7. Index

8. Special Symbols and Notations