Spectral correspondences for Maaß Waveforms on Quaternion Groups

Why is $N^{new}_{\Gamma_0(12)}(\lambda)$ of cocompact type?

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(Joint with Stefan Lemurell)

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Georgia

Medgar Evers College

- Established in 1969.
- Medgar Wiley Evers (1925-1963)
- Central Brooklyn community.
- Senior college–City University of New York System.
- Create and sustain, local, research driven, mathematical communities

Outline

1. Some motivating historical ideas
   - Spectroscopy
   - Hearing the Shape of a drum
2. What did we prove?
3. An outline of the proof & Speculation

Auguste Comte

Figure 1: Auguste Comte, 1798–1857.

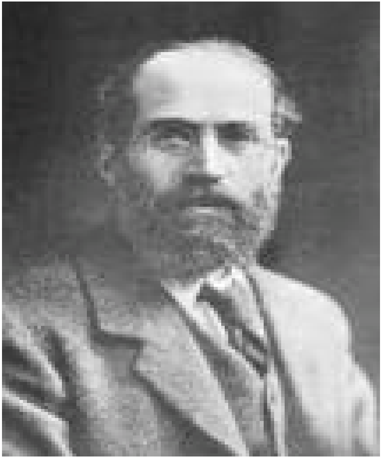
We understand the possibility of determining their shapes, their distances, their sizes and their movements; whereas we would never know how to study by any means their chemical composition, or their mineralogical structure, and, even more so, the nature of any organized beings that might live on their surface (Comte, 1835).

Key Questions Emerging

1. Given knowledge of the structure of an atom or molecule, could one predict the discrete set of vibration frequencies of the system?
2. Conversely, given the spectrum of a vibrating system, what can be inferred about the system's structure?

Arthur Schuster

To find out the different tunes sent out by a vibrating system is a problem which may or may not be solvable in certain special cases, but it would baffle the most skillful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out. (Schuster, 1882)



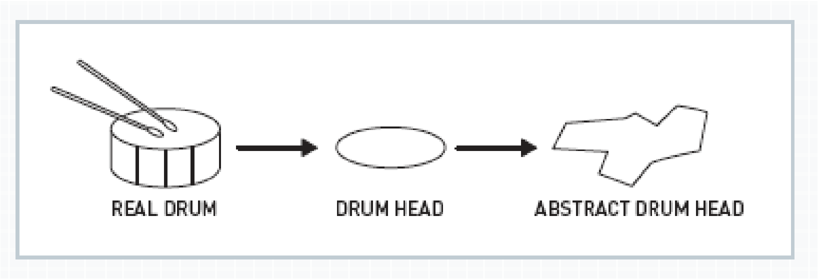
Mark Kac

In 1966 Mark Kac drew the attention of mathematicians to spectral problems by posing a question that is a prototype for those arising in spectral theory: Can we hear the shape of a drum?

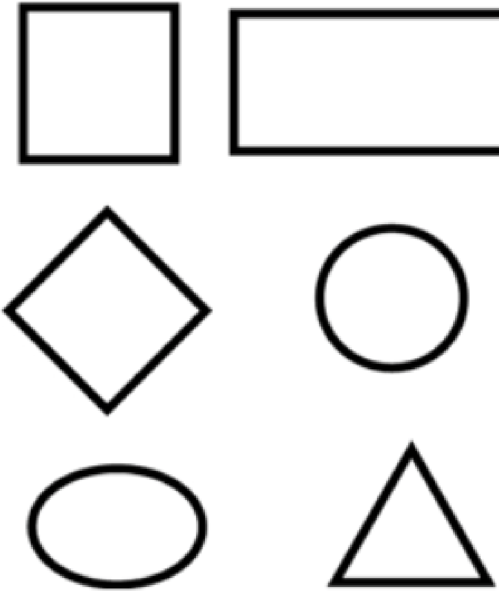
Can you hear the shape of a drum?

- Given the shape, what can we infer about the sound; and
- Given the sound, can we reconstruct the shape?

From real drums to mathematical drums

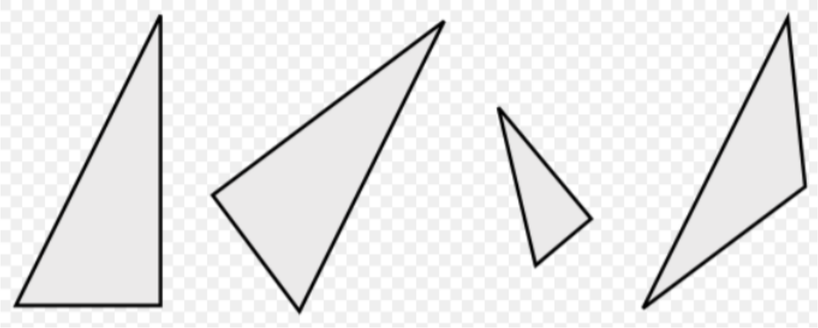


Some “simple” drums



Congruent Triangles

- Evidently two congruent domains must have the same spectrum, we say that the domains are isospectral.
- Does isospectrality guarantee congruence?



Can we hear the shape of a triangle among all triangles?

Theorem (Grieser & Maronna)

A triangle is determined uniquely up to congruence by its area A, its perimeter P, and the sum R of the reciprocals of its angles.

Corollary

One can hear the shape of a triangle among all triangles. That is, if we know that Ω is a triangle, then the spectrum of Ω determines which triangle it is.

Isospectral Manifolds

- John Milnor, 1964–an isospectral pair of flat tori of 16 dimensions which are not isometric
- M. F. Vignéras, 1980–based on the Selberg trace formula for $PSL_2(\mathbb{R})$ and $PSL_2(\mathbb{C})$ constructed examples of isospectral, non-isometric closed hyperbolic 2-manifolds and 3-manifolds as quotients of hyperbolic 2-space and 3-space by arithmetic subgroups

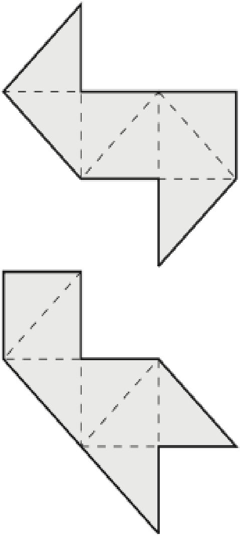
The Sunada Method(1985)

...a general method of constructing isospectral pairs based on a covering space technique

the idea:

- M is a finite covering of a compact Riemannian manifold M_0
- G the finite group of deck transformations
- H_1, H_2 are subgroups of G meeting each conjugacy class of G in the same number of elements,
- then the manifolds $H_1 \backslash M$ and $H_2 \backslash M$ are isospectral but not necessarily isometric

Isospectral Drums



What can we say about an isospectral set of planar regions?...an isospectral set of compact Riemannian manifolds?

Theorem (Sarnak, Phillips, Osgood(1988))

(A) *An isospectral set of closed Riemannian two manifolds is compact in the* $\mathcal{C}^\infty$ *topology.*

(B) *An isospectral set of planar drums is compact in the* $\mathcal{C}^\infty$ *topology.*

Problem

Are planar domains isospectrally rigid; i.e., is every one parameter family of isospectral planar domains necessarily an isometric family?

Risager

Asymptotic Densities of Maaß Newforms

For which Hecke congruence groups $\Gamma_0(M)$ is $N^{new}_{\Gamma_0(M)}(\lambda)$, i.e., the spectral counting function of Laplace eigenvalues for Maass newforms on $\Gamma_0(M)$, of cocompact type?

Spectral Theory–a sketch

(i) (a) $\mathcal{H}$
    (b) Γ
    (c) $\Gamma\backslash\mathcal{H}$

(ii) $\Delta_\Gamma = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

(iii) $L^2\left(\Gamma\backslash\mathcal{H}\right)$

(i) f is an eigenfunction of Δ_Γ,

(ii) λ is an eigenvalue of Δ_Γ.

(iii) f is a **Maaß waveform**

$f \in L^2\left(\Gamma\backslash\mathcal{H}\right)$

(i) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$.

(ii) f vanishes at the cusps of Γ and

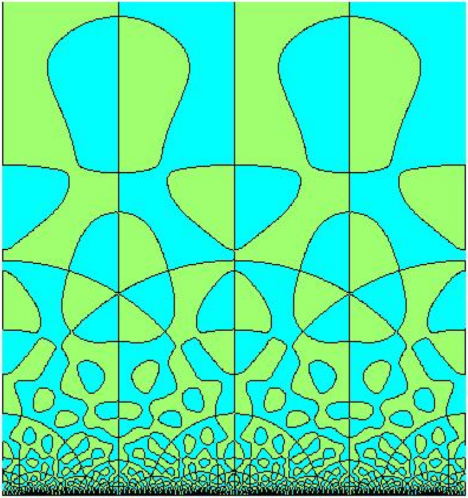
(iii) $\Delta_\Gamma f = \lambda f$ for some $\lambda > 0$

(i) basis for $L^2\left(\Gamma\backslash\mathcal{H}\right)$ on $\Gamma\backslash\mathcal{H}$.

(ii) no explicit construction exists for any of these functions

(iii) number theory, dynamical systems and quantum chaos

A display of a Maas Waveform computed for the Hecke Congruence subgroup $\Gamma_0(7)$



The spectrum of Δ_Γ.

- $\Delta_{\mathcal{O}^1}$ on $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ is discrete, and is comprised of the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \ , \lambda_n \longrightarrow \infty.$$

- $\Delta_{\Gamma_0(M)}$ on $L^2\left(\Gamma_0(M)\backslash\mathcal{H}\right)$ has both a continuous spectrum $\left[\frac{1}{4}, \infty\right)$ and a discrete spectrum contained in $[0, \infty)$. The discrete eigenvalues satisfy

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \ , \mu_n \longrightarrow \infty.$$

The spectral counting function $N_\Gamma(\lambda)$

for a cofinite group Γ is defined as follows:

$$N_\Gamma(\lambda) = \#\{\lambda_n \leq \lambda : \lambda_n \in dSpec(\Delta_\Gamma)\}.$$

It has asymptotic expansion of form:

(a) $N_\Gamma(\lambda) = \frac{Area(\Gamma\backslash\mathcal{H})}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda})$,when Γ is a co-compact group; and

(b) $N_\Gamma(\lambda) = \frac{Area(\Gamma\backslash\mathcal{H})}{4\pi}\lambda + O(\sqrt{\lambda}\log\lambda)$, when Γ is a non co–compact but co–finite group;

(c) We note the difference in error terms in the co–compact and the non–cocompact cases

Newforms

Let $a, m, d \in \mathbb{N}$ such that $m < d$ and $am|d$.

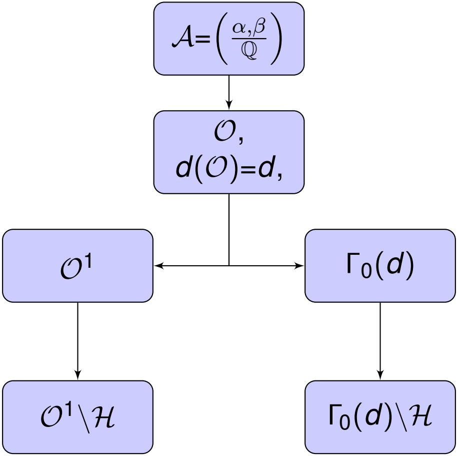
(i) If $m|d$ then $\Gamma_0(d) \subset \Gamma_0(m)$.

(ii) If $f(z)$ is a Maaß form on $\Gamma_0(m)$ then $f(az)$ is a Maaß form on $\Gamma_0(d)$ for all $a|\frac{d}{m}$.

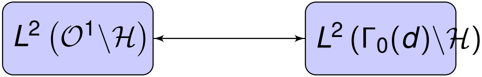
(iii) Such functions are called "oldforms" on $\Gamma_0(d)$ and it is natural to avoid such functions in a search for Maaß forms, for they naturally belong on the larger group $\Gamma_0(m)$.

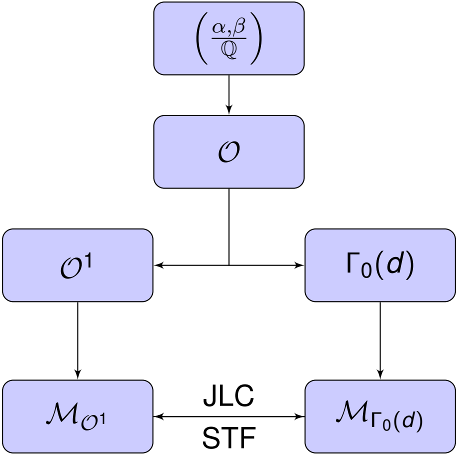
(v) The Maaß forms which naturally live on $\Gamma_0(d)$ are called "newforms".

The Basic Outline



$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$





$\left(\frac{\alpha,\beta}{\mathbb{Q}}\right)$
$\mathcal{O}$
$\mathcal{O}^1$
$\Gamma_0(d)$
$\mathcal{M}_{\mathcal{O}^1}$
JLC
STF
$\mathcal{M}_{\Gamma_0(d)}$

Asymptotic densities of Maaß Newforms,

(i) Γ-cofinite arithmetic Fuchsian group;
   (a) $\Gamma_c = \mathcal{O}^1$
   (b) $\Gamma_{nc} = \Gamma_0(d)$

(ii) Δ_Γ-automorphic Laplacian related to Γ

(iii) $N_\Gamma(\lambda) = \#\{\lambda_n \leq \lambda : \lambda_n \in Spec(\Delta_\Gamma)\}$-corresponding spectral counting function

(i) $N_{\mathcal{O}^1}(\lambda)$ has an asymptotic expansion of the form:

$$N_{\mathcal{O}^1}(\lambda) = \frac{Vol(\mathcal{O}^1 \backslash \mathcal{H})}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log \lambda}).$$

(ii) $N_{\Gamma_0(d)}(\lambda)$ has an asymptotic expansion of the form:

$$N_{\Gamma_0(d)}(\lambda) = \frac{Vol(\Gamma_0(d) \backslash \mathcal{H})}{4\pi}\lambda + O(\sqrt{\lambda}\log \lambda).$$

(iii) Difference in error terms: cocompact-$O(\frac{\sqrt{\lambda}}{\log \lambda})$ and non-cocompact $O(\sqrt{\lambda}\log \lambda)$

$N^{new}_{\Gamma_0(d)}(\lambda)$ of Cocompact Type,

(i) $N^{new}_{\Gamma_0(d)}(\lambda)$-counts only the eigenvalues corresponding to newforms on $\Gamma_0(d)$.

(ii) when d is the product of an even number of different primes:

$$N^{new}_{\Gamma_0(d)}(\lambda) = c_d \frac{Vol(\Gamma_0(d)\backslash\mathcal{H})}{4\pi}\lambda + O\left(\frac{\sqrt{\lambda}}{\log\lambda}\right)$$

where $0 < c_d < 1$.

(iii) This is an expansion of form characteristic of the cocompact case.

- How is this explained?
- Are there other values of d for which $N^{new}_{\Gamma_0(d)}(\lambda)$ is of this form?

Are there other values of d for which $N^{new}_{\Gamma_0(d)}(\lambda)$ is of this form?

Theorem (Risager)

Let $M, n, t \in \mathbb{N}$ be the natural numbers defined uniquely by the requirements that n be squarefree and that $M = nt^2$. $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type if and only if at least one of the following holds:

(i) *n contains at least two primes.*

(ii) *n is a prime and $4||M$.*

Are there spectral correspondences which explain the Theorem?

How is this explained?

(i) Correspondences between spaces of automorphic forms for cocompact(i.e., compact quotient space)(Γ_c) and non-cocompact (Γ_{nc}) arithmetic Fuchsian groups.

(ii) Jacquet Langlands Correspondence–To any nonconstant eigenfunction of the Laplacian on a cocompact arithmetic Fuchsian group there corresponds a nontrivial cuspform with the same eigenvalue on some non-cocompact but cofinite arithmetic Fuchsian group.

Spectral theory of automorphic Laplacians

(i) The adelic-representation theoretic view point or
(ii) The classical perspective of the upper half-plane

Adelic-Representation theoretic

(i) Jacquet and Langlands, 1970
(ii) Hideo Shimizu, 1972

Classical

(i) Eichler, A. Selberg, 1950's.
(ii) Hejhal, 1983
(iii) Bolte and Johansson: 1999
(iv) Strömbergsson, 2000 and
(v) Risager, 2003
(vi) Blackman, 2011, Blackman & Lemurell, 2014

Hejhal

A classical approach to a well-known spectral correspondence on quaternion groups

(i) correspondence can be established using completely classical techniques.

(ii) integral transform
$\Theta : \mathcal{M}_{\mathcal{O}^1} \subset L_0^2\left(\mathcal{O}^1\backslash\mathcal{H}\right) \hookrightarrow \mathcal{C}_{\Gamma_0(M)} \subset L^2\left(\Gamma_0(M)\backslash\mathcal{H}\right)$.

Matters Arising

(i) the precise range of Θ?

(ii) was Θ injective?

Suggestions for Solutions

(i) space of $\Gamma_0(N)$-newforms

(ii) Selberg trace formula .

Bolte and Johannson

Extended Hejhal's work

(i) *Theta-lifts of Maaß waveforms*,

(ii) *A Spectral Correspondence for Maaß waveforms*

Theorem (Bolte and Johannson)

All eigenvalues of the hyperbolic Laplacian on $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ *also occur as eigenvalues of the hyperbolic Laplacian on* $L^2\left(\Gamma_0(M)\backslash\mathcal{H}\right)$ *where* $M = d(\mathcal{O})$.

Theta lifts

(i) preserve eigenvalues of the hyperbolic Laplacian

(ii) they can be extended to arbitrary orders

(iii) Natural link-$\mathcal{M}_{\mathcal{O}^1} \longleftrightarrow \mathcal{C}_{\Gamma_0(M)}$ where $M = discriminant(\mathcal{O})$

Theorem (Bolte and Johansson)

Let $\mathcal{O}^1 \subset \mathcal{O}_{max} \subset \mathcal{A}$ be group of units of norm one a maximal order in $\mathcal{A}$ with discriminant$(\mathcal{O}_{max}) = M$. Then the positive Laplace eigenvalues, including multiplicities, on $\mathcal{O}^1 \backslash \mathcal{H}$ coincide with the Laplace spectrum on $\mathcal{M}^{new}_{\Gamma_0(M)}$.

I.e., Laplace eigenvalues and their multiplicities for the cocompact group $\mathcal{O}^1$ coincide with those for the newforms of level M . I.e., in the language of the Selberg Trace formula-

$$\sum_{r_k \in Spec\left(\Delta_{\mathcal{O}^1_{max}}\right)} h(r_k) = \sum_{r_k \in Spec\left(\Delta^{new}_{\Gamma_0(M)}\right)} h(r_k) \qquad (3.1)$$

for an arbitrary test function h.

Bolte and Johansson(Lemurell)-Matters arising

(i) Did this result imply that Θ provided a bijection between Laplace eigenspaces in $L^2(\mathcal{O}^1\backslash\mathcal{H})$ and $\mathcal{M}^{new}_{\Gamma_0(d)} \subset L^2(\Gamma_0(d)\backslash\mathcal{H})$?

(ii) yes. with the assumption of one dimensionality of the relevant eigenspaces-without this assumption one could not even assert that the $Range(\Theta) \in \mathcal{M}^{new}_{\Gamma_0(d)}$

(iii) Strömbergsson's Thesis

Strömbergsson

(i) Choose a Hecke basis of Maaß waveforms $\varphi_1, \varphi_2, \varphi_3, \ldots$ in $\mathcal{M}_{\mathcal{O}^1} \subset L^2(\mathcal{O}^1\backslash\mathcal{H})$; ordered with increasing eigenvalues. Write: $\Delta\varphi_k + \lambda_k\varphi_k = 0$ for some $0 = \lambda_0 \leq \lambda_1 \leq \lambda_3 \leq \ldots$

(ii) $g_1, g_2, g_3, \ldots$ be a basis of newforms in the newspace $\mathcal{M}^{new}_{\Gamma_0(d)} \subset \mathcal{M}_{\Gamma_0(d)}$; ordered with increasing eigenvalues. We write: $\Delta g_k + \mu_k g_k = 0$ for some $0 = \mu_0 \leq \mu_1 \leq \mu_3 \leq \ldots$

(iii) integral transform $\Theta : \mathcal{M}_{\mathcal{O}^1} \subset L^2_0\left(\mathcal{O}^1\backslash\mathcal{H}\right) \hookrightarrow \mathcal{M}_{\Gamma_0(d)} \subset L^2\left(\Gamma_0(d)\backslash\mathcal{H}\right)$.

Theorem (Strömbergsson)

Given unit group $\mathcal{O}^1$ and Hecke congruence subgroup $\Gamma_0(d)$, as above. We have $\lambda_k = \mu_k$ for all $k \geq 1$. In particular, for any $\lambda > 0$, Θ is a bijection from $\mathcal{M}_{\mathcal{O}^1}(\lambda)$ onto $\mathcal{M}^{new}_{\Gamma_0(d)}(\lambda)$.

What values of M are explained thus far?

Case I
(a) n -product of an even number of primes
(b) $t^2 = 1$

In this case we take $\mathcal{O}^1$ to be a unit group in a maximal order $\mathcal{O}$ in an indefinite rational quaternion division algebra $\mathcal{A}$ with $d_{\mathcal{A}} = n$. This correspondence is described classically by Strömbergsson and Bolte and Johansson. I.e.,$\mathcal{M}_{\mathcal{O}^1}(\lambda) \longleftrightarrow \mathcal{M}^{new}_{\Gamma_0(d)}(\lambda)$

Case II (a) n -product of an even number of primes

(b) $t^2 \neq 1$, with $(n, t^2) = 1$

We take $\mathcal{O}^1 \subset \mathcal{O}_{t^2} \subset \mathcal{O}_{max} \subset \mathcal{A}$ with $d_{\mathcal{A}} = n$ This correspondence is described classically by both Risager and Strömbergsson. They prove

Theorem (Strömbergsson and Risager)

If $\mathcal{O}^1$ is the unit group in an Eichler order of level t^2 in a maximal order $\mathcal{O} \subset \mathcal{A}$ with reduced discriminant $d_{\mathcal{A}} = n$ then there is a correspondence given by an integral operator such that for $\lambda \neq 0$ a certain $\lambda-$ new eigenspace of $\Delta_{\mathcal{O}^1}$ is in bijection with the $\lambda-$new eigenspace of $\Delta_{\Gamma_0(M)}$ where $M = nt^2$.

Case III (a) $n = pq$, p, q − *primes* > 2, $p \neq q$
(b) $t^2 = p^{2r}q^{2s}$, $r, s \geq 1$

We take $\mathcal{O}^1_{pq,p^{2r}q^{2s}} \subset \mathcal{O}_{pq,p^{2r}q^{2s}} \subset \mathcal{O}_{pq} \subset \mathcal{A}$-with $e(\mathcal{O}_{pq,p^{2r}q^{2s}})_p = e(\mathcal{O}_{pq,p^{2r}q^{2s}})_q = -1$ and $d_{\mathcal{A}} = pq$. $\Longrightarrow$ $d_{\mathcal{O}_{pq,p^{2r}q^{2s}}} = p^{2r+1}q^{2s+1}$-

$\mathcal{M}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}(\lambda) \longleftrightarrow \mathcal{M}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}(\lambda)$

Using the Selberg trace formula, we show that the positive Laplace eigenvalues, including multiplicities, for Maaß newforms on $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$ coincides with the Laplace spectrum on Maaß newforms for the Hecke congruence group $\Gamma_0(p^{2r+1}q^{2s+1})$ where $r, s \geq 1$. I.e. $\mathcal{M}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$ corresponds with $\mathcal{M}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$

Theorem (Blackman & Lemurell)

Assume that r *is a positive integer that is divisible by an even number of primes, and that every prime dividing* r *does so to an odd power. Let* u *be any positive integer relatively prime to* r*. Then the positive Laplace eigenvalues, including multiplicities, for Maaß newforms on* $\mathcal{O}^1(r, u)$ *and* $\Gamma_0(ru)$ *coincide.*

Correspondence between Maaß newforms on orders in different quaternion algebras.

Corollary

Assume that r_1 and r_2 are positive integers each divisible by an even number of primes, and that every prime dividing r_1 or r_2 does so to an odd power. Let u_1 and u_2 be any positive integers relatively prime to r_1 and r_2 respectively such that $r_1 u_1 = r_2 u_2$. Then the positive Laplace eigenvalues, including multiplicities, for Maaß newforms on $\mathcal{O}^1(r_1, u_1)$ and $\mathcal{O}^1(r_2, u_2)$ coincide.

The smallest example matching the corollary is

$$\mathcal{M}^{new}_{\mathcal{O}^1_{6,5}} = \mathcal{M}^{new}_{\mathcal{O}^1_{10,3}} = \mathcal{M}^{new}_{\mathcal{O}^1_{15,2}}$$

Conjecture

In the case $N = 4pu^2$ with p a prime and u odd there is no quaternion order with a natural correspondence between newforms as the one in our Theorem.

Our Strategy
What does the comparison via the Selberg trace formulas entail?

1. A special linear combination of $\Gamma_0(ru)$-trace formulas has the property that parabolic term vanishes, and
   1.1 identity(area of fundamental domain),
   1.2 elliptic, and
   1.3 hyperbolic

   terms all agree with the corresponding terms for a linear combination of $\mathcal{O}^1$-trace formulas
2. Local embedding numbers
3. The agreement of these local factors which gives us the correspondence.

The Selberg Trace Formula

1. Selberg trace formula: establish a spectral correspondence between $\mathcal{M}^{new}_{\Gamma_0(p^3q^3)}$ and $\mathcal{M}^{new}_{\mathcal{O}^1_{pq,p^2q^2}}$.
2. What is the STF? general identity connecting geometrical and spectral terms, i.e., an identity of form:

$$\sum \text{spectral terms} = \sum \text{geometric terms}. \qquad (4.1)$$

The spectral terms come from discrete and continuous spectrum of the automorphic hyperbolic Laplacian Δ_Γ for a cofinite Fuchsian group Γ

The geometrical terms are integral operators depending on the conjugacy classes of Γ.

We will need two versions of it

1. 1.1 for the cocompact groups $\mathcal{O}^1$
   1.2 for the Hecke congruence groups $\Gamma_0(m)$.
2. The trace formula for both types of these Fuchsian groups under consideration are well-known
3. We recall the known results

In what follows $h: \mathbb{C} \to \mathbb{C}$ satisfying

1. $h(r) = h(-r)$,
2. $h(r)$ is holomorphic in the strip $|\Im(r)| \leq \frac{1}{2} + \varepsilon$, for some $\varepsilon > 0$,
3. $|h(r)| \leq \frac{C}{(1+\Re(r))^{2+\delta}}$ for some $C > 0$ and $\delta > 0$.

The Selberg Trace Formula-$\mathcal{O}^1$

1. The Fourier transform of h will then be written as

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r)\, e^{-iru}\, dr\ .$$

2. Let $\lambda_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(\mathcal{O}^1 \backslash \mathcal{H})$, counted with multiplicities.
3. Then

$$\begin{aligned}
\sum_{k=0}^{\infty} h(r_k) = & \frac{A_{\mathcal{O}^1}}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\, dr \\
& + \sum_{t\in\{0,1\}} \frac{E'(t,1,\mathcal{O}^1)}{2m_t} \sum_{k=1}^{m_t-1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\, dr \\
& + \sum_{t=3}^{\infty} E'(t,1,\mathcal{O}^1)\, \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)}\ .
\end{aligned} \tag{4.2}$$

We will view equation above as having form

$$\sum_{k=0}^{\infty} h(r_k) = \mathcal{I} + \mathcal{E} + \mathcal{H} \tag{4.3}$$

where

$$\mathcal{I}_{\mathcal{O}^1} = \frac{A_{\mathcal{O}^1}}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\, dr \tag{4.4}$$

will denote the identity contribution,

$$\mathcal{E}_{\mathcal{O}^1} = \sum_{t\in\{0,1\}} \frac{E'(t,1,\mathcal{O}^1)}{2m_t} \sum_{k=1}^{m_t-1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\, dr \tag{4.5}$$

the elliptic contribution and

$$\mathcal{H}_{\mathcal{O}^1} = \sum_{t=3}^{\infty} E'(t,1,\mathcal{O}^1)\, \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)} \tag{4.6}$$

the hyperbolic contribution. We recall that in the case of cocompact groups there is no continuous spectrum and no parabolic element.

The Selberg Trace Formula-Hecke Congruence Groups-$\Gamma_0(m)$

Let $\mu_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $\Gamma_0(m)\backslash\mathcal{H}$, counted with multiplicities. Then

$$\begin{aligned}\sum_{k=0}^{\infty} h(r_k) =& \frac{A_m}{4\pi}\int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\, dr \\ &+ \sum_{t\in\{0,1\}} \frac{E'(t,1,\Gamma_0(m))}{2m_t} \sum_{k=1}^{m_t-1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\, dr \\ &+ \sum_{t=3}^{\infty} E'(t,1,\Gamma_0(m))\, \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k\operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k\operatorname{arccosh}(\frac{t}{2})\right)} \\ &+ 2^{\omega(m)}\Big\{ \hat{h}(0)\log(\tfrac{\pi}{2}) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r) \left[\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\right] dr \\ &\quad + 2\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}\, \hat{h}(2\log n) - \sum_{\substack{p\mid m\\ p \text{ prime}}} \sum_{k=0}^{\infty} \frac{\log p}{p^k}\, \hat{h}(2k\log p) \Big\}. \end{aligned} \tag{4.7}$$

$$\sum_{k=0}^{\infty} h(r_k) = \mathcal{I} + \mathcal{E} + \mathcal{H} + \mathcal{P} \tag{4.8}$$

where

$$\mathcal{I}_{\Gamma_0(m)} = \frac{A_m}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\, dr \tag{4.9}$$

$$\mathcal{E}_{\Gamma_0(m)} = \sum_{t\in\{0,1\}} \frac{E'(t,1,\Gamma_0(m))}{2m_t} \sum_{k=1}^{m_t-1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\, dr \tag{4.10}$$

$$\mathcal{H}_{\Gamma_0(m)} = \sum_{t=3}^{\infty} E'(t,1,\Gamma_0(m))\, \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)} \tag{4.11}$$

$$\mathcal{P}_{\Gamma_0(m)} = 2^{\omega(m)} \Bigg\{ \hat{h}(0) \log(\tfrac{\pi}{2}) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r) \left[\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir) \right] d$$

$$+ 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}\, \hat{h}(2\log n) - \sum_{\substack{p\mid m \\ p \text{ prime}}} \sum_{k=0}^{\infty} \frac{\log p}{p^k}\, \hat{h}(2k \log p) \tag{4.12}$$