# $\mathcal{A}$-Calculus on a Real Associative Algebra

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# a word on algebras and representations

1. **Algebra:** $\mathcal{A}$ denotes a real $n$-dimensional vector space with an associative unital multiplication.
2. **Right $\mathcal{A}$-linear maps:** If $T : \mathcal{A} \to \mathcal{A}$ is a linear transformation with $T(v \star w) = T(v) \star w$ then $T \in \mathcal{L}_{\mathcal{A}}$. Since $1 \in \mathcal{A}$ we find:

$$T(v) = T(1 \star v) = T(1) \star v = L_{T(1)}(v)$$

3. **Regular representation:** If $M = [T]_\beta$ for $T \in \mathcal{L}_{\mathcal{A}}$ then $M \in \mathrm{M}_{\mathcal{A}}(\beta)$.
4. **Algebra Isomorphisms:** $\mathcal{A} \simeq \mathcal{L}_{\mathcal{A}} \simeq \mathrm{M}_{\mathcal{A}}(\beta)$. For $\mathcal{A} = \mathbb{R}^n$:

$$\alpha \longleftrightarrow L_\alpha \longleftrightarrow [\alpha \star e_1 | \alpha \star e_2 | \cdots | \alpha \star e_n]$$

5. **Application to Quaternions:**

$$a + bi + cj + dk \longleftrightarrow L_{a+bi+cj+dk} \longleftrightarrow \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

# differentiability over an algebra

## Definition

Let $U \subseteq \mathcal{A}$ be an open set containing $p$. If $f : U \to \mathcal{A}$ is a function then we say $f$ is $\mathcal{A}$-**differentiable at** $p$ if there exists a linear function $d_pf \in \mathcal{L}_{\mathcal{A}}$ such that

$$\lim_{h \to 0} \frac{f(p+h) - f(p) - d_pf(h)}{||h||} = 0.$$

## Theorem (The $\mathcal{A}$-Cauchy Riemann Equations)

*If $f : \mathcal{A} \to \mathcal{A}$ is $\mathcal{A}$-differentiable at $p$ and $\mathcal{A}$ has basis $\{v_1, \dots, v_n\}$ with $v_1 = 1$ and coordinates $x_1, \dots, x_n$ then $\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_1} \star v_k$ for $k = 2, \dots, n$.*

**Proof:** by definition $d_pf(v_k) = \frac{\partial f}{\partial x_k}$. If $f$ is $\mathcal{A}$-differentiable at $p$ then $d_pf$ exists and is right $\mathcal{A}$-linear and $d_pf(v_k) = d_pf(1) \star v_k$ thus $\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_1} \star v_k$. $\square$

Observation: there are $n^2 - n$ generalized-CR equations

# concerning differentiability via difference quotients

## Definition

Let $f : \text{dom}(f) \to \mathcal{A}$ be a function where $\text{dom}(f)$ is open and $p \in \text{dom}(f)$.

**(1.)** If $f$ is $\mathcal{A}$-differentiable at $p$ then $f$ is $D_1$ at $p$. .

**(2.)** If $\displaystyle\lim_{\mathcal{A}^\times \ni \zeta \to p} \frac{f(\zeta) - f(p)}{\zeta - p}$ exists then $f$ is $D_2$ at $p$.

## Theorem

*Let $U$ be an open set in $\mathcal{A}$. If $f$ is $D_2$ at each point in $U$ then $f$ is $D_1$ on $U$.*

However, in $\mathcal{A} = \mathbb{R} \oplus \epsilon\mathbb{R}$ with $\epsilon^2 = 0$ the function $f(\zeta) = \zeta$ is **nowhere** $D_2$.

## Theorem

*Let $U$ be an open set in a commutative semisimple finite dimensional real algebra $\mathcal{A}$. The set of $D_1$ functions on $U$ coincides with the set of $D_2$ functions on $U$.*

# connecting the $\mathcal{A}$-CR equations and $\text{M}_{\mathcal{A}}$

- Let $\mathcal{A} = \mathbb{C}$. If $f = u + iv \in \text{C}_{\mathcal{A}}(\mathcal{A})$ then $\text{J}_f = \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix}$. Note, $\text{J}_f \in \text{M}_{\mathcal{A}}$ implies $\text{J}_f = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ thus $\underbrace{\partial_x u = \partial_y v \ , \ \partial_x v = -\partial_y u}_{\text{CR eqs.}}$.

- If $\mathcal{A} = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ then $\text{M}_{\mathcal{A}}$ has matrices with form $\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$. If $f = u + jv + j^2 w \in \text{C}_{\mathcal{A}}(\mathcal{A})$ then six CR eqns. follow:

$$\text{J}_f = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = \begin{bmatrix} u_x & w_x & v_x \\ v_x & u_x & w_x \\ w_x & v_x & u_x \end{bmatrix} \Rightarrow \underbrace{\begin{array}{c} u_x = v_y = w_z \\ v_x = w_y = u_z \\ w_x = u_y = v_z \end{array}}_{\text{CR eqs.}}$$

# differential $\mathcal{A}$-calculus:

Suppose that $f$ and $g$ are $\mathcal{A}$-differentiable,

1. $\dfrac{d}{d\zeta}(f+g) = \dfrac{df}{d\zeta} + \dfrac{dg}{d\zeta}$
2. $d_p(f \star g)(h) = d_pf(h) \star g(p) + f(p) \star d_pg(h)$, and if $\mathcal{A}$ commutative,
$$\frac{d}{d\zeta}(f \star g) = \frac{df}{d\zeta} \star g + f \star \frac{dg}{d\zeta}$$
3. $\dfrac{d}{d\zeta}\left[f(g(\zeta))\right] = \dfrac{df}{d\zeta}(g(z)) \star \dfrac{dg}{d\zeta}$
4. $\mathcal{A}$ commutative, let $f(\zeta) = \zeta^n$ for $n \in \mathbb{N}$ then $f'(\zeta) = n\zeta^{n-1}$
5. Let $\Psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism of unital, associative finite dimensional algebras over $\mathbb{R}$. If $f$ is $\mathcal{A}$ differentiable at $p$ then $g = \Psi \circ f \circ \Psi^{-1}$ is $\mathcal{B}$-differentiable at $\Psi(p)$. Moreover, $g'(p) = (\Psi \circ f' \circ \Psi^{-1})(p)$.

# Taylor's Theorem

## Definition

Let $U \subseteq \mathcal{A}$ be an open set and $f : U \to \mathcal{A}$ an $\mathcal{A}$-differentiable function on $U$ then we define $f' : U \to \mathcal{A}$ by $f'(p) = (d_p f)(1)$ for each $p \in U$. Higher derivatives are defined in the usual fashion: $f^{(k+1)}(p) = (d_p f^{(k)})(1)$.

## Theorem

*If $f : \mathcal{A} \to \mathcal{A}$ is $k$-times $\mathcal{A}$-differentiable then*

$$\frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} = \frac{\partial^k f}{\partial x_1^k} \star v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k}.$$

## Theorem

*Let $\mathcal{A}$ be a commutative, unital, associative algebra over $\mathbb{R}$. If $f$ is real analytic at $p \in \mathcal{A}$ then $f(p+h) = f(p) + f'(p) \star h + \frac{1}{2} f''(p) \star h^2 + \cdots + \frac{1}{k!} f^{(k)}(p) \star h^k + \cdots$ where $h^2 = h \star h$ and $h^{k+1} = h^k \star h$ for $k = 1, 2, \dots$.*

# on how PDEs arise from $\mathcal{A}$-differentiability

This was shown by Wagner in his 1948 Thesis, and an improved proof was given by Waterhouse in 1992. This can be extended to higher order:

**Theorem**

*Let $U$ be open in $\mathcal{A}$ and suppose $f : U \to \mathcal{A}$ is twice $\mathcal{A}$-differentiable on $U$. If there exist $B_{ij} \in \mathbb{R}$ for which $\sum_{i,j} B_{ij} v_i \star v_j = 0$ then $\sum_{i,j} B_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$.*

**Proof:** suppose $f$ is twice continuously $\mathcal{A}$-differentiable on $U \subset \mathcal{A}$ and suppose there exist $B_{ij} \in \mathbb{R}$ for which $\sum_{i,j} B_{ij} v_i \star v_j = 0$. Multiply by $\frac{\partial^2 f}{\partial x_1^2}$ to obtain:

$$\sum_{i,j} B_{ij} \frac{\partial^2 f}{\partial x_1^2} \star v_i \star v_j = 0 \quad \Rightarrow \quad \sum_{i,j} B_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = 0.$$

## Example ($\mathcal{A}$-Laplace eqns for $\mathcal{A} = \mathbb{C}$ are the usual Laplace Equations)

Consider $\mathcal{A} = \mathbb{R} \oplus i\mathbb{R}$ where $i^2 = -1$. Notice, we have multiplication table and Hessian matrix

| | 1 | $i$ |
|---|---|---|
| 1 | 1 | $i$ |
| $i$ | $i$ | $-1$ |

&

| | $x$ | $y$ |
|---|---|---|
| $x$ | $f_{xx}$ | $f_{xy}$ |
| $y$ | $f_{yx}$ | $f_{yy}$ |

If $f = u + iv$ then $f_{xx} = -f_{yy}$, provides $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

## Example ($\mathcal{A}$-Laplace eqns for 3-hyperbolic numbers)

Consider $\mathcal{A} = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ where $j^3 = 1$. Notice, we have multiplication table and Hessian matrix

| | 1 | $j$ | $j^2$ |
|---|---|---|---|
| 1 | 1 | $j$ | $j^2$ |
| $j$ | $j$ | $j^2$ | 1 |
| $j^2$ | $j^2$ | 1 | $j$ |

&

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | $f_{xx}$ | $f_{xy}$ | $f_{xz}$ |
| $y$ | $f_{yx}$ | $f_{yy}$ | $f_{yz}$ |
| $z$ | $f_{zx}$ | $f_{yz}$ | $f_{zz}$ |

Hence, $f_{xx} = f_{yz}$, $f_{xy} = f_{zz}$, $f_{xz} = f_{yy}$

# an algebra for the wave equation

Consider the speed $c$ wave equation $c^2 u_{xx} = u_{tt}$. Let us find an algebra $\mathcal{W}_c$ which takes the speed-$c$ wave equation as its generalized Laplace Equation. Let $(x, t) = x + kt$ form a typical point in the algebra. What rule should we give to $k$? Consider:

$$c^2 u_{xx} = u_{tt} \quad \leftrightarrow \quad c^2 = k^2$$

thus set $k^2 = c^2$. The algebra $\mathcal{W}_c = \mathbb{R} \oplus k\mathbb{R}$ with $k^2 = c^2$ has $\mathcal{W}_c$-differentiable functions $f = u + kv$ for which $c^2 u_{xx} = u_{tt}$. Note $\mathcal{W}_c \approx \mathbb{R} \times \mathbb{R}$ is given by:

$$\Phi(x + kt) = (x + ct, x - ct)$$

If $F = (F_1, F_2)$ is $(\mathbb{R} \times \mathbb{R})$-differentiable then $F_1 = F_1(x_1), F_2 = F_2(x_2)$ and every $\mathcal{W}_c$-differentiable function is connected to such $F$; $f = \Phi^{-1} \circ F \circ \Phi$.

$$\boxed{f(x + kt) = \frac{1}{2}(F_1(x + ct) + F_2(x - ct)) + \frac{k}{2c}(F_1(x + ct) - F_2(x - ct))}$$

d'Alembert's solution appears naturally in the function theory of $\mathcal{W}_c$

# Conjugate Variables for Commutative $\mathcal{A}$

Alvarez-Parrilla, Frías-Armenta, López-González and Yee-Romero (PAGR)(2012),

## Definition (conjugate variables)

Suppose $\mathcal{A}$ has an invertible basis $\beta = \{v_1, v_2, \dots, v_n\}$ with $v_1 = 1$. If $\zeta = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$ then we define the $j$-**th conjugate of** $\zeta$ as follows:

$$\bar{\zeta}_j = \zeta - 2x_j v_j = x_1 + \cdots + x_{j-1} v_{j-1} - x_j v_j + x_{j+1} v_{j+1} + \cdots + x_n v_n$$

for $j = 2, 3 \dots, n$.

However, I differ from (PAGR) in my construction of $\frac{\partial}{\partial \zeta}$

## Definition (Wirtinger Derivatives for $\mathcal{A}$)

Suppose $f : \mathcal{A} \to \mathcal{A}$ is $\mathbb{R}$-differentiable. Furthermore, suppose $\beta = \{1, v_2, \dots, v_n\}$ is an invertible basis and $\zeta = x_1 + x_2 v_2 + \cdots + x_n v_n$. We define

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left( (3-n)\frac{\partial}{\partial x_1} + \frac{1}{v_2}\frac{\partial}{\partial x_2} + \cdots + \frac{1}{v_n}\frac{\partial}{\partial x_n} \right) \quad \& \quad \frac{\partial}{\partial \bar{\zeta}_k} = \frac{1}{2}\left( \frac{\partial}{\partial x_1} - \frac{1}{v_k}\frac{\partial}{\partial x_k} \right)$$

for $j = 2, 3, \dots, n$.

# Wirtinger Calculus for Commutative $\mathcal{A}$

## Theorem

$$\frac{\partial \zeta}{\partial \zeta} = 1, \quad \frac{\partial \bar{\zeta}_j}{\partial \zeta} = 0, \quad \frac{\partial \bar{\zeta}_j}{\partial \bar{\zeta}_j} = 1, \quad \frac{\partial \bar{\zeta}_j}{\partial \bar{\zeta}_k} = 0, \quad \frac{\partial \zeta}{\partial \bar{\zeta}_j} = 0$$

*for all $j = 2, \dots, n$ and $k \neq j$. Note: PAGR had $\frac{n-2}{n}$ instead of 0*

## Theorem

*Let $\beta = \{1, v_2, \dots, v_n\}$ be an invertible basis for the commutative algebra $\mathcal{A}$. If $f : \mathcal{A} \to \mathcal{A}$ is $\mathcal{A}$-differentiable at $p$ then $\dfrac{\partial f}{\partial \bar{\zeta}_j} = 0$ for $j = 2, \dots, n$.*

**Proof:**

$$\frac{\partial f}{\partial \bar{\zeta}_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \frac{1}{v_k} \star \frac{\partial f}{\partial x_k}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \frac{1}{v_k} \star \frac{\partial f}{\partial x_1} \star v_k\right) = 0$$

as $\mathcal{A}$ is assumed commutative and $\frac{1}{v_k} \star v_k = 1$. $\square$

# a nowhere holomorphic function on $\mathcal{A}$

## Example

Suppose $\dim(\mathcal{A}) \geq 2$. Let $f(\zeta) = \zeta\,\bar{\zeta}_2$ where $f : \mathcal{A} \to \mathcal{A}$ then

$$\frac{\partial f}{\partial \zeta} = \frac{\partial \zeta}{\partial \zeta}\bar{\zeta}_2 + \zeta\frac{\partial \bar{\zeta}_2}{\partial \zeta} = \bar{\zeta}_2 \qquad \& \qquad \frac{\partial f}{\partial \bar{\zeta}_2} = \frac{\partial \zeta}{\partial \bar{\zeta}_2}\bar{\zeta}_2 + \zeta\frac{\partial \bar{\zeta}_2}{\partial \bar{\zeta}_2} = \zeta$$

This function is only $\mathcal{A}$-differentiable at the origin. In the usual complex analysis it is simply the square of the modulus; $f(z) = z\bar{z} = x^2 + y^2$ where $z = x + iy$ has $\bar{z}_2 = x - iy$.

# Laplace Equations for 3-hyperbolic numbers

Consider $\mathcal{A} = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ where $j^3 = 1$, $\zeta = x + jy + zj^2$ and $\bar{\zeta}_2 = x - jy + j^2 z$ and $\bar{\zeta}_3 = x + jy - j^2 z$.

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left[j\frac{\partial}{\partial y} + j^2\frac{\partial}{\partial z}\right] \quad \& \quad \frac{\partial}{\partial \bar{\zeta}_2} = \frac{1}{2}\left[\frac{\partial}{\partial x} - j^2\frac{\partial}{\partial y}\right] \quad \& \quad \frac{\partial}{\partial \bar{\zeta}_3} = \frac{1}{2}\left[\frac{\partial}{\partial x} - j\frac{\partial}{\partial z}\right].$$

$$\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\frac{\partial}{\partial z} = 2\left(\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_2} + \frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial^2}{\partial \bar{\zeta}_2^2}\right)$$

$$\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial z}\frac{\partial}{\partial x} = 2j^2\left(-2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_2} + \frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_3} - \frac{\partial}{\partial \bar{\zeta}_2}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial^2}{\partial \bar{\zeta}_3^2}\right)$$

$$\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial x}\frac{\partial}{\partial y} = 2j\left(-2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_2} - \frac{\partial}{\partial \bar{\zeta}_2}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial^2}{\partial \bar{\zeta}_2^2}\right)$$

If $f = u + vj + j^2 w$ is an $\mathcal{A}$-differentiable function then $\frac{\partial f}{\partial \bar{\zeta}_2} = 0$ and $\frac{\partial f}{\partial \bar{\zeta}_3} = 0$. It follows that the component functions of $f$ must solve the corresponding PDEs:

$$\Phi_{xx} - \Phi_{yz} = 0, \quad \Phi_{yy} - \Phi_{zx} = 0, \quad \Phi_{zz} - \Phi_{xy} = 0.$$

These are the **generalized Laplace Equations** for the 3-hyperbolic numbers.

# how to change a real PDE to an $\mathcal{A}$-ODE (sometimes)

**Theorem**

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\bar{\zeta}_2} + \cdots + \frac{\partial}{\bar{\zeta}_n} \quad \textit{and} \quad \frac{\partial}{\partial x_k} = v_k\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\bar{\zeta}_2} + \cdots + \frac{\partial}{\bar{\zeta}_n} - 2\frac{\partial}{\bar{\zeta}_k}\right)$$

**(1.)** given a PDE in real independent variables $x_1, x_2, \dots, x_n$ choose an algebra $\mathcal{A}$ of dimension $n$ to study in conjunction with the system.

**(2.)** convert the derivatives in the PDE with respect to $x_1, x_2, \dots, x_n$ to derivatives with respect to the algebra variables $\zeta, \bar{\zeta}_2, \dots, \bar{\zeta}_n$

**(3.)** impose that the derivatives with respect to $\bar{\zeta}_2, \dots, \bar{\zeta}_n$ vanish, study the resulting ordinary differential equation in $\zeta$. If possible, solve the $\mathcal{A}$-ODE which results.

# integration on commutative $\mathcal{A}$

**Theorem**

*If $\zeta : [t_o, t_1] \to \mathcal{A}$ is differentiable parametrization of a curve $C$ and $f$ is continuous near $C$ then* $\int_C f(\zeta) \star d\zeta = \int_{t_o}^{t_f} f(\zeta(t)) \star \frac{d\zeta}{dt}\, dt.$

For $\mathcal{A}$ with norm-$\| \cdot \|$ there exists $m_{\mathcal{A}} > 0$ such that $\|x \star y\| \leq m_{\mathcal{A}} \|x\| \|y\|$

**Theorem**

*Let $C$ be a rectifiable curve with arclength $L$. Suppose $||f(\zeta)|| \leq M$ for each $\zeta \in C$ and suppose $f$ is continuous near $C$. Then*

$$\left|\left| \int_C f(\zeta) \star d\zeta \right|\right| \leq m_{\mathcal{A}} ML$$

*where $m_{\mathcal{A}}$ is a constant such that $||z \star w|| \leq m_{\mathcal{A}} ||z|| \, ||w||$ for all $z, w \in \mathcal{A}$.*

# theorems of integration on commutative $\mathcal{A}$

## Theorem

*Suppose* $f = \frac{dF}{d\zeta}$ *near a curve* $C$ *which begins at* $P$ *and ends at* $Q$ *then*

$$\int_C f(\zeta) \star d\zeta = F(Q) - F(P).$$

## Theorem

*Let* $f : U \to \mathcal{A}$ *be a function where* $U$ *is a connected subset of* $\mathcal{A}$ *then the following are equivalent:*

**(i.)** $\int_{C_1} f \star d\zeta = \int_{C_2} f \star d\zeta$ *for all curves* $C_1, C_2$ *in* $U$ *beginning and ending at the same points,*

**(ii.)** $\int_C f \star d\zeta = 0$ *for all loops in* $U$*,*

**(iii.)** $f$ *has an antiderivative* $F$ *for which* $\frac{dF}{d\zeta} = f$ *on* $U$*.*

# theorems of integration on on commutative $\mathcal{A}$

## Theorem

*Let $f : U \to \mathcal{A}$ be a function where $U$ is a simply connected subset of commutative $\mathcal{A}$ and suppose $f$ is continuously differentiable in the real Frechet sense. The $\mathcal{A}$-valued one-form $f \star d\zeta$ is exact if and only if $f$ is $\mathcal{A}$-differentiable.*

**Proof:** Let $d\zeta = v_1 dx_1 + \cdots + v_n dx_n$ and note $\mathcal{A}$ commutative gives $d\zeta \wedge d\zeta = 0$. Suppose $f$ is $\mathcal{A}$-differentiable:

$$\begin{aligned} d(f \star d\zeta) &= df \wedge d\zeta \\ &= (\partial_1 f\, dx_1 + \partial_2 f\, dx_2 + \cdots \partial_n f\, dx_n) \ \wedge d\zeta \\ &= (\partial_1 f dx_1 + \partial_1 f \star v_2 dx_2 + \cdots + \partial_1 f \star v_n dx_n) \ \wedge d\zeta \\ &= \partial_1 f \star d\zeta \wedge d\zeta \\ &= 0. \end{aligned}$$

Thus $f \star d\zeta$ is exact by Poincare's Lemma. Conversely, if there exists $\phi$ for which $d\phi = f \star d\zeta$ and it follows $f \star v_j = \partial_j \phi$ and as $v_1 = 1$ we find $f = \partial_1 \phi$ thus

$$\partial_j f = \partial_j \partial_1 \phi = \partial_1 \partial_j \phi = \partial_1 (f \star v_j) = (\partial_1 f) \star v_j$$

Thus $f$ is $\mathcal{A}$-differentiable. □

# theorems of integration on commutative $\mathcal{A}$

## Theorem (Cauchy's Integral Theorem)

*If $U \subseteq \mathcal{A}$ is simply connected then $\int_C f \star d\zeta = 0$ for all loops $C$ in $U$ if and only if $f$ is $\mathcal{A}$-differentiable on $U$.*

## Theorem (FTC part I for $\mathcal{A}$)

*Let $C$ be a differentiable curve from $\zeta_o$ to $\zeta$ in $U \subseteq \mathcal{A}$ where $U$ is an open simply connected subset of $\mathcal{A}$. Assume $f$ is $\mathcal{A}$ differentiable on $U$ then*

$$\frac{d}{d\zeta} \int_C f(\eta) \star d\eta = f(\zeta).$$

## Theorem (real smooth and $\mathcal{A}$-differentiable imply $\mathcal{A}$-smooth)

*Let $\mathcal{A}$ be a commutative unital finite dimensional algebra over $\mathbb{R}$. Suppose $f : \mathcal{A} \to \mathcal{A}$ has arbitrarily many continuous real derivatives at $p$ and suppose $f$ is once $\mathcal{A}$-differentiable at $p$ then $f^{(k)}(p)$ exists for all $k \in \mathbb{N}$.*

**Proof:** Suppose $\mathcal{A}$ has $\beta = \{1, \dots, v_n\}$. Assume inductively that $f^{(k)}(p)$ exists hence $f^{(k)}(p) = \frac{\partial^k f(p)}{\partial x_1^k}$. Consider, omitting $p$ to reduce clutter,

$$\frac{\partial f^{(k)}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\frac{\partial^k f}{\partial x_1^k}\right] = \frac{\partial^k}{\partial x_1^k}\left[\frac{\partial f}{\partial x_j}\right] = \frac{\partial^k}{\partial x_1^k}\left[\frac{\partial f}{\partial x_1}\right] \star v_j = \frac{\partial f^{(k)}}{\partial x_1} \star v_j.$$

Thus $f^{(k)}$ is $\mathcal{A}$-differentiable at $p$ which proves $f^{(k+1)}(p)$ exists. $\square$

## Theorem

*If $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and $\zeta_k$ serves as $\mathcal{A}_k$ variable for $k = 1, 2$. If $F : \mathcal{A} \to \mathcal{A}$ is $\mathcal{A}$-differentiable then $F = (F_1, F_2)$ where $F_1 = F_1(\zeta_1)$ and $F_2 = F_2(\zeta_2)$*

If $\mathcal{A}$ picks up $\mathbb{R}$ as a factor then $\mathcal{A}$ cannot have a Cauchy's Integral Formula.

# Further Results with Daniel Freese on $\mathcal{A}$

Here we mostly assume $\mathcal{A}$ is commutative.

- Theory of sequences and series including Cauchy criterion, product theorem, modified ratio and root tests etc.
- Theory of power series over $\mathcal{A}$. Weird behaviour as regards to zero divisors, but, for units almost the usual story. Weierstrauss $M$-test for normal convergence.
- Theory of elementary functions; exponential, sine, cosine, sinh and cosh.
- If $\mathcal{A}$ is generated by $k$ with $k^n = \pm 1, 0$ then

$$e^{kx} = \cos(x) + k \sin_{1,n}(x) + \cdots + \sin_{n,n-1} k^{n-1}$$

  and we derive an identity $\det(M(e^{kx}) = 1$ which includes $\cos^2 x + \sin^2 x = 1$ and $\cos^2 x + \sin^2 x = 1$. More exciting, is $e^{jx} = l(x) + jm(x) + j^2 c(x)$ where $l^3 + m^2 + c^3 - 3lmc = 1$ for the 3-hyperbolic numbers. Many open questions remain in the theory of special functions over $\mathcal{A}$
- Details to be found in *Theory of Series in the A-calculus and the n-Pythagorean Theorem* in preparation (to appear on ArXiV soon)

# Further Results with Nathan BeDell on $\mathcal{A}$

Here we mostly assume $\mathcal{A}$ is commutative with $\dim(\mathcal{A}) = N$

- (BeDell) further analysis of Freese's special functions. Also a theory of logarithms for a large class of associative, unital algebras over $\mathbb{R}$.
- theory of uniform convergence, term-by-term $\mathcal{A}$-integration and $\mathcal{A}$-differentiation theorems. Existence and uniqueness theorem and general solution set for $\mathcal{A}$-ODE. Proof uses usual Picard iteration adapted to the $\mathcal{A}$-integral
- theory of Wronskian for $n$-th order linear $\mathcal{A}$-ODE including the $n$-th order variation of parameter formula and Abel's Formula
- solution to general $n$-th order constant $\mathcal{A}$-coefficient $\mathcal{A}$-ODE. For $P(x) \in \mathcal{A}[x]$ we find $P(D)[y] = 0$ has exponential solutions $e^{\lambda\zeta}$ corresponding to factor $D - \lambda$ in $P(D)$. However, some $P(D)$ are not split-linear over $\mathcal{A}$ hence we use $\mathcal{A}' = \mathcal{A}[k]/\langle P(k) \rangle$ with $\zeta \mapsto e^{k\zeta}$ and we show the $\mathcal{A}$-component functions are solutions to $P(D)[y] = 0$.
- solution to $A\vec{\zeta} = \frac{d\vec{\zeta}}{dt}$ for any constant $\mathcal{A}$-valued matrix $A$ via matrix exponential and e-vectors.
- Details to be found in *Ordinary Differential Equations over Associative Algebras* and *Logarithms and Trigonometric Functions over Associative Algebras* in preparation (to appear on ArXiV soon)

# quaternions and $\mathcal{A}$-differentiability.

Consider $f(\zeta) = \zeta^2$ for quaterion variable $\zeta = t + ix + jy + kz$. Calculate,

$$f(\zeta) = t^2 - x^2 - y^2 - z^2 + 2t(ix + jy + kz)$$

We find:

$$J_f = \begin{bmatrix} 2t & -2x & -2y & -2z \\ 2x & 2t & 0 & 0 \\ 2y & 0 & 2t & 0 \\ 2z & 0 & 0 & 2t \end{bmatrix} \text{ vs. } \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \in \text{M}(\mathbb{H})$$

For $\zeta \neq 0$, we see $J_f$ is **not** in the regular representation of $\mathbb{H}$.

**Theorem**

*Right $\mathbb{H}$-linear functions are $\mathbb{H}$-differentiable functions on $\mathbb{H}$.*

**Proof:** let $f(\zeta) = \alpha\zeta$ then $f(\zeta + h) = \alpha(\zeta + h) = f(\zeta) + \alpha h$. Hence $df(h) = \alpha h$ and $df(hk) = \alpha(hk) = df(h)k$.

# differentiability over an algebra

Let $\mathcal{A} = \mathbb{R}^6$ with the following noncommutative multiplication:

$$(a,b,c,d,e,f) \star (x,y,z,u,v,w) = (ax, by, cz, au+dy, bv+ez, aw+dv+fz)$$

The regular representation of $\mathcal{A}$ has typical element

$$\mathbf{M}(a,b,c,d,e,f) = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & d & 0 & a & 0 & 0 \\ 0 & 0 & e & 0 & b & 0 \\ 0 & 0 & f & 0 & d & a \end{bmatrix}$$

Suppose $\mathcal{A}$ has variables $\zeta = (x_1, \dots, x_6)$ and define $f(\zeta) = (1,1,1,1,1,x_3^2)$ and define $g(\zeta) = (0,0,0,x_2,0,x_5)$. Calculate

$$(f \star g)(\zeta) = (0,0,0,x_2,0,x_5) \quad \& \quad (g \star f)(\zeta) = (0,0,0,x_2,0,x_2+x_5)$$

Observe $f$ and $g$ are $\mathcal{A}$-differentiable and $f \star g = g$ is likewise $\mathcal{A}$-differentiable. In contrast, $g \star f$ is not $\mathcal{A}$-differentiable as its Jacobian matrix is nonzero in the $(2,6)$-entry. ($\mathcal{A}$ here is not simple, Rosenfeld's result does not apply)

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- THANKS! Questions ?