A categorical approach to the Cayley-Dickson construction

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Game Plan

The Cayley-Dickson construction takes one from the reals to the complex numbers, from the complex to the quaternions, from the quaternions to the octonions, from the octonions to . . . etc. Each time a certain property of the algebraic structure is lost. A sketch is a categorical way of describing algebraic structures. We define a functor from the category of sketches to the category of sketches that takes a description of an algebraic structure and outputs the description of the Cayley-Dickson construction on that structure. With this we will have a clearer way of seeing what properties are lost with the Cayley-Dickson construction.

Game Plan

- Some number systems.
- The Cayley-Dickson construction
- Sketches
- Morphisms of sketches
- Cayley-Dickson as a functor of sketches
- An open and related question

Some Number Systems

- Real numbers $\mathbb{R}$
- Complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$. No longer totally ordered.
- Quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. No longer commutative.
- Octonins $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}k$. No longer associative.
- Sedenions $\mathbb{S} = \mathbb{O} \oplus \mathbb{O}m$. No longer a normed division algebra.
- 32 dim.... No longer alternative.
- etc.

The Cayley-Dickson Construction

- Start with a real field $\mathbb{F}$ and look at the real field $\mathbb{F} \times \mathbb{F}$. The addition and subtraction is defined "point-wise".
- The multiplication is defined as a

$$(a, b)(c, d) = (ac - db^*, a^*d + cb).$$

- The involution is $(a, b)^* = (a^*, -b)$.

Dealing with General Structures

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- Universal algebra
- Model theory
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There are different fields of mathematics that discuss general structures:

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We are going to work with category theory because they discuss functions from one structure to another structure. Within category theory there are several ways of talking about general structures:

- Triples (or Monads)
- Algebraic theories
- Sketches

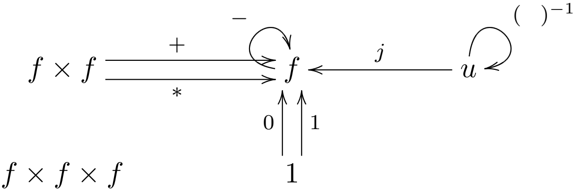
We work with sketches because we can get to the nittly-gritty details of how it works. Also, sketches are more powerful.

Sketches

- There is a whole hierarchy of types of sketches. We are going to focus on **finite discrete sketches**.
- A sketch $S = (G, D, P, C)$ is a graph G, a set of diagrams D (equations) in the graph, a set of products P in the graph, and a set of coproducts C in the graph.
- A model of a sketch S in a category $\mathbf{C}$ is a graph homomorphism $F : G \longrightarrow \mathbf{C}$ such that the diagrams in D go to commutative diagrams in $\mathbf{C}$, the product's in P go to products in $\mathbf{C}$, and the coproducts in C go to coproducts in $\mathbf{C}$.
- A morphism of models from $F : G \longrightarrow \mathbf{C}$ to $F' : G \longrightarrow \mathbf{C}$ is simply a natural transformation from F to F'.
- For a given sketch S and a category $\mathbf{C}$, the collection of models and morphisms form a category $Mod(S, \mathbf{C})$.

The Sketch of Fields

- From “Categories for Computer Science” by Barr and Wells.
- The objects of the graph are $1, u, f, f \times f, f \times f \times f$.
- The edges of the graph are



- The diagrams are
  - associativity of $+$ and $*$.
  - commutativity of $+$ and $*$.
  - 0 is the additive unit, 1 is the multiplicative unit.
  - $-$ is the additive inverse, $(\ \ )^{-1}$ is the multiplicative inverse.
  - $*$ distributes over $+$.
- The products are 1, $f \times f$ and $f \times f \times f$
- The coprducts are $0 : 1 \longrightarrow f \longleftarrow u : j$

Morphisms of Sketches and the Category of Sketches

- A morphism from sketch $S = (G, D, P, C)$ to sketch $S' = (G', D', P', C')$ is a graph homomorphism $H : G \longrightarrow G'$ such that if $d : I \longrightarrow G$ is a diagram in D, then $H \circ d : I \longrightarrow G'$ is a diagram in D'. H also takes products in P to products in P', and coproducts in C to coproducts in C'.
- The collection of sketches and sketch morphisms form a category **Sketch**.
- A sketch homomorphism $H : S \longrightarrow S'$ induces a functor $H^* : Mod(S', \mathbf{C}) \longrightarrow Mod(S, \mathbf{C})$ where $F : G' \longrightarrow \mathbf{C}$ goes to $F \circ H : G \longrightarrow G' \longrightarrow \mathbf{C}$.

Cayley-Dickson as a functor of sketches

$$CD : \mathbf{Sketch} \longrightarrow \mathbf{Sketch}$$

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This does not work. We cannot do the Cayley-Dickson construction on any sketch. It has to have a multiplication, addition, subtraction and an involution, etc.

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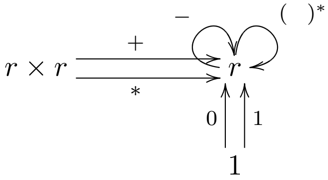
We want the smallest structure so that we can still do the CD construction. Its like a ring with an involution but leave out the commutative and the associative. It is a magma form of a ring with an involution.
We call this sketch MRI (Magma Ring with Involution) and look at the slice category $MRI/\mathbf{Sketch}$ which consists of sketches that contain MRI as a subsketch ("sketches over MRI").

$$CD : MRI/\mathbf{Sketch} \longrightarrow MRI/\mathbf{Sketch}$$

Cayley-Dickson as a functor of sketches

The MRI sketch has the following structure.

- The objects of the graph are $1, r, r \times r$.
- The edges of the graph are



- The diagrams are
  - associativity of $+$.
  - commutativity of $+$.
  - 0 is the additive unit, 1 is the multiplicative unit.
  - $-$ is the additive inverse.
  - $*$ distributes over $+$.
  - $(\ \ )^*$ is its own inverse.
- The products are 1, $r \times r$
- The coprducts

Cayley-Dickson as a functor of sketches

Given a sketch $S = (G, D, P, C)$ over MRI,

$$f : MRI \longrightarrow S$$

we apply the functor CD to get

$$CD(f) : MRI \longrightarrow CD(S) = S'$$

where

$$S' = (G \times G, D \times D, P \times P, C \times C).$$

Cayley-Dickson as a functor of sketches

Whereas

$$MRI \xrightarrow{CD(f)} S'$$

$$\begin{array}{ccc} r \times r & & f \times f \times f \times f \\ \Big\downarrow + & & \Big\downarrow \langle 1,3,2,4 \rangle \\ r & & f \times f \times f \times f \\ & & \Big\downarrow f(+)\times f(+) \\ & & r \times r \end{array}$$

$$\begin{array}{ccc} r & & f \times f \\ \Big\downarrow - & & \Big\downarrow f(-)\times f(-) \\ r & & f \times f \end{array}$$

Cayley-Dickson as a functor of sketches

The multiplication and involution works as follows

$$MRI \xrightarrow{\quad CD(f) \quad} S'$$

$$\begin{array}{ccc}
r \times r & & f \times f \times f \times f \\
\Big\downarrow{\scriptstyle *} & & \Big\downarrow{\scriptstyle \Delta} \\
r & & f^{\times 8} \\
\end{array}$$

$$\begin{array}{ccc}
f^{\times 8} & & f \times f \\
\Big\downarrow{\scriptstyle \langle 1,3,4,2,5,8,7,6 \rangle} & & \Big\uparrow{\scriptstyle +\times-} \\
f^{\times 8} & \xrightarrow[*\times*(id,()^*)\times*(()^*,id)\times*]{} & f^{\times 4}
\end{array}$$

$$\begin{array}{ccc}
r & & f \times f \\
\Big\downarrow{\scriptstyle (\quad)^*} & & \Big\downarrow{\scriptstyle f((\quad)^*)\times f(-)} \\
r & & f \times f
\end{array}$$

Structure of the Cayley Dickson functor.

- There is a natural transformation

$$\eta : CD \Longrightarrow Id_{MRI/\mathbf{Sketch}}.$$

- This induces η^* which takes every $\mathbb{F}$ to $\mathbb{F} \oplus \mathbb{F}i$ i.e. $f \mapsto (f, 0)$.
- There is a natural transformation

$$\alpha : CD \circ CD \Longrightarrow CD.$$

- This induces α^* which every number system $\mathbb{F}$, takes $\mathbb{F} \oplus \mathbb{F}i$ to $(\mathbb{F} \oplus \mathbb{F}i) \oplus (\mathbb{F} \oplus \mathbb{F}i)$ which takes (f, f') to $(f, f', 0, 0)$
- α is associative.
- This is "kind of" a monad on $MRI/\mathbf{Sketch}$ and a "kind of" a comonad on semantics.

An open and related question.

For any particular algebraic structure there are many different ways of describing the structure. For example: a group can be described as

- $* : G \times G \longrightarrow G$, $' : G \longrightarrow G$, $e \in G$;

$$x * (y * z) = (x * y) * z, \qquad x * (x') = e = (x') * x, \qquad e * x = x$$

- $* : G \times G \longrightarrow G$ and $' : G \longrightarrow G$

$$((z * (x * y)')' * (z * y')) * (y' * y)' = x$$

- $* : G \times G \longrightarrow G$ and $' : G \longrightarrow G$

$$y * (z * (((w * w') * (x * z)') * y))' = x$$

- $/ : G \times G \longrightarrow G$;

$$x/((((x/x)/y)/z)/(((x/x)/x)/z)) = y$$

An open and related question.

Question: Is there a measure μ on signatures such that
If signature S_x describes the same structure as signature S_y (i.e., S_x and S_y are Morita equivalent),
then $\mu(S_x) = \mu(S_y)$
???

An open and related question.

- Losing structure: $\mu(CD(S)) < \mu(S)$.
- μ does not take values in natural numbers, rather in real numbers because:

$$\mu(MRI) < \cdots < \mu(CD^3(S)) < \mu(CD(CD(S))) < \mu(CD(S)) < \mu(S).$$

- Levels of structure

$$\mu() < \mu(x(yz)_3 =_3 (xy)z)$$

$$< \mu(x(yz)_3 =_3 (xy)z; xy_2 =_2 yx)$$

$$< \mu(x(yz)_3 =_3 (xy)z; xy_2 =_2 yx; x^*{}_1 =_1 x)$$

- (I think this has to do with Kolmogorov complexity theory).

Thank You!

References

- Baez, John C., “The octonions”, Bull. Amer. Math. Soc. 39), Pgs. 145-205, 2002.
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